Solutions to HW1, Stat 401 Spring 2011

(1). (a) Here $S = X_1 + \cdots + X_{100}$, where the X_i random variables are independent identically distributedf (*iid*) with density $f(x) = 2x$ for $0 \le x \le 1$. Therefore

$$E(S) = E(X_1) + \cdots + E(X_{100}) = 100\,E(X_1) \quad , \qquad \mathrm{Var}(S) \;=\; 100\,\mathrm{Var}(X_1)$$

where

$$E(X_1) \;=\; \int_0^1 x\,2x\,dx \;=\; (2x^3/3)\Big|_0^1 = 2/3$$

$$\mathrm{Var}(X_1) \;=\; E(X_1^2) - (2/3)^2 \;=\; \int_0^1 x^2\,2x\,dx \;-4/9 \;=\; 1/2 - 4/9 = 1/18$$

The Central Limit Theorem says we should treat S as being approximately normal with its proper mean and variance, i.e., $S \approx \mathcal{N}(200/3,\, 50/9)$. Therefore the probabilities in (a) are (to three decimal places) 0.240, 0.760, 0, 0, where for example the second one is figured as

$$P(65 < S < 75) \;=\; P\Big(\frac{65 - 66.667}{\sqrt{50/9}} < \frac{S - 66.667}{\sqrt{50/9}} < \frac{75 - 66.667}{\sqrt{50/9}}\Big)$$

$\approx\; \Phi(3.536) \;-\; \Phi(-.707) \;=\; 0.760$.

(b). Here $N = \sum_{i=1}^{100} I_{[X_i > 0.6]} \sim \mathrm{Binom}(100, p)$ with mean $100p$ and variance $100p(1-p)$, where $p = P(X_1 > 0.6) \;=\; \int_{.6}^1 2x\,dx \;=\; 0.64$.

(2).=Sim.3 (a) Here $n = 1000$, $P(U_i > 0.6) = 0.4$, so $N = \sum_{i=1}^{1000} I_{[U_i > 0.6]} \sim \mathrm{Binom}(1000, .4)$.

(b). So by the Law of Large Numbers, with high probability $N/1000 \approx E(I_{[U_1 > 0.4]}) = 0.4$.

(c) Here also the law of large numbers says the average is close with high probability to the expectation $E(e^{-U_1}) \;=\; \int_0^1 e^{-u}\,du \;=\; 1 - e^{-1}$.

(d). The distribution function is $F_{\exp(-U)}(t)\,P(e^{-U} \le t) \;=\; P(U \ge (-log(t))) \;=\; 1 + \log(t)$ for $0 < t < 1/e$. The corresponding density, on the same interval, is $f(t) \;=\; F'(t) \;=\; 1/t$.

(3). (a) $E(S_1) = (1/20)\,40\,(\vartheta/2) \;=\; \vartheta$ which shows that S_1 is unbiased for ϑ.

(b). As suggested in the Hint: for $0 < t < \vartheta$, $F_{S_2}(t) = P(S_2 \le t) = P(X_1 \le t,\, X_2 \le t,\, \ldots, X_{40} \le t) = (P(X_1 \le t))^{40} = (t/\vartheta)^{40}$, which implies that the density of S_2 is $f(t) = F'_{S_2}(t) = 40t^{39}/\vartheta^{40}$ for $0 < t < \vartheta$. Therefore

$$E(S_2) \;=\; \frac{40}{\vartheta^{40}} \int_0^{\vartheta} t^{40}\, dt \;=\; \frac{40\,\vartheta^{41}}{41\,\vartheta^{40}} \;=\; \frac{40}{41}\,\vartheta$$

from which it follows immediately that $(41/40)\, S_2$ is an unbiased estimator for ϑ.

(4). (a) Since the variance of a Uniform$[0,\vartheta]$ random variable U_i is $\vartheta^2/12$, we conclude

$$\text{std.err}(S_1) \;=\; \Big(\text{Var}(S_1)\Big)^{1/2}\Big|_{\vartheta=S_1} \;=\; \frac{1}{20}\Big(40\,(S_1)^2/12\Big)^{1/2} \;=\; S_1/\sqrt{30}$$

(b) Using the density for S_2 above, we find

$$\text{Var}(S_2) \;=\; \frac{40}{\vartheta^{40}} \int_0^{\vartheta} t^{41}\, dt \;-\; (40\vartheta/41)^2 \;=\; \vartheta^2\,(40/42 - (40/41)^2)$$

So $\text{std.err}(41S_2/40) = (\frac{41}{40}\, S_2)/\sqrt{40 \cdot 42}$. (Note that the second unbiased estimator has much smaller standard error than the first !)

(5). Here $n = 1000$, and $P(0 < Y_1 \le 1/2) = P(1/2 < Y_1 \le 1) = 1/6$, $P(1 < Y_1 \le 3/2) = P(3/2 < Y_1 \le 2) = 1/3$, so $N_1, N_2 \sim$ Binom$(1000, 1/6)$, $N_3, N_4 \sim$ Binom$(1000, 1/3)$, and

$$E(\frac{N_1}{1000}) \;=\; E(\frac{N_2}{1000}) \;=\; \frac{1}{6}\,, \quad E(\frac{N_3}{1000}) \;=\; E(\frac{N_4}{1000}) \;=\; \frac{1}{3}$$

$$\text{Var}(\frac{N_1}{1000}) \;=\; \text{Var}(\frac{N_2}{1000}) \;=\; \frac{1}{6000}\,, \quad \text{Var}(\frac{N_3}{1000}) \;=\; \text{Var}(\frac{N_4}{1000}) \;=\; \frac{1}{3000}$$