Solutions to HW3, Stat 401 Spring 2011

Note: since #38(b) on p.277 is a tolerance interval, a topic I said we skipped, the 5 points for that problem are extra-credit. So the denominator for this HW is 75 points.

(1). You were asked to exhibit CI's for 100 batches of Gamma(1.3,2.6) of size 40.Code and plots are given below using ideas from `Rscripts/Confint.RLog`.

(a). The calculations for the three intervals are as follows:

```
tmp = array(rgamma(4000,1.3,2.6), c(100,40))   ### 100 batches
xbar = c(tmp %*% rep(1/40,40))
SD = apply(tmp,1,sd)
CUp.Int = xbar+1.96*SD/sqrt(40); CLo.Int = xbar-1.96*SD/sqrt(40)
twidth = qt(.975,39)*SD/sqrt(40)
```

(b)
```
plot(1:100, CUp.Int[1:100], col="blue", ylim=c(0,.8))
points(1:100, CLo.Int[1:100], col="red")
abline(h=0.5)
for(i in 1:100) lines(c(i,i), c(CLo.Int[i],CUp.Int[i]),
        lty=3, col="brown")
```
In my example, 5 intervals failed to cover 0.5: 4 too low, 1 too high.

(c). If the intervals were working perfectly, then the number of times μ_0 falls outside the interval out of 100 should be $\text{Binom}(100, .05) \approx \text{Poisson}(5)$, which has expected value 5 and probability of ≤ 1 equal to $pbinom(1, 100, .05) = 0.04$, and probability of ≥ 9 equal to $1 - pbinom(8, 100, .05) = .065$. So numbers of 1 or fewer or 9 or more are a little unlikely.

(2),#27 p.270 plus extra R steps. Two examples of intervals of these types for various $(n, k) = (78, 40)$ and $(77, 39)$, are as follows:

```
> c(CI7.Q=CI7.10(40,78), CI7.N=CI7.11(40,78), CIp27 = CI7.10(42,82))
   CI7.Q1    CI7.Q2    CI7.N1    CI7.N2    CIp271    CIp272
0.4039270 0.6205105 0.4018959 0.6237451 0.4059077 0.6173910
> c(CI7.Q=CI7.10(39,77), CI7.N=CI7.11(39,77), CIp27 = CI7.10(41,81))
   CI7.Q1    CI7.Q2    CI7.N1    CI7.N2    CIp271    CIp272
0.3972001 0.6151698 0.3948236 0.6181634 0.3995078 0.6122788
```

The first three are very close to one another, as are the second three. But we combine all possible k's, using their appropriate probability weights, in calculating coverage probabilities through the function `Cover` described in the `Confint.RLog` script. For the requested (n, p) combinations, this gives:

```
> Cover(78,.57)
        CI7.10    CI7.11     CIp27
[1,] 0.9566156 0.9358466 0.9566156     ## CIp27 and CI7.10 best
> Cover(47,.53)
      CI7.10   CI7.11     CIp27
[1,] 0.934841 0.934841 0.9590036       ## CIp27 best
> Cover(46,.16)
        CI7.10    CI7.11     CIp27
[1,] 0.9424594 0.9592895 0.8931543   ## CI7.10 best, CIp27 awful!
```

Finally, to give examples of slightly different n's which give very different ordering for the closeness of these intervals' coverage probabilities outcomes to .95, consider:

```
 > Cover(77,.57)
       CI7.10    CI7.11     CIp27    ## CI7.10 and CI7.11 equally good
[1,] 0.9409148 0.9409148 0.9585606    ## CIp27 may be slightly better

> Cover(51,.47)
   CI7.10    CI7.11     CIp27
0.9315672 0.9224678 0.9585676        ### CIp27 clearly best.
> Cover(43,.17)
   CI7.10    CI7.11     CIp27
0.9459912 0.9611444 0.8985123     ### CI7.10 best; CIp27 still awful.
```

(3), # 22 p. 269. The 99% one-sided upper-bounding confidence interval for unknown binomial proportion p based on $\hat{p} = X/n = 0.072$ for $n = 487$ is (0, 0.0991 according to formula (7.11) and (0, 0.1041) based on formula (7.10). The difference is not large, but that latter should be regarded as slightly more accurate. (Either one is sufficient for full credit on the exercise, with 2 points extra for those who compared both.)

(4), #26, p. 269. In this problem $\bar{X} = 4.06$. The 2-sided $1-\alpha$ level confidence interval consists of all values λ between the two roots of the quadratic equality $(\bar{X}-\lambda)^2 - z^2_{\alpha/2}\,(\lambda/n) = 0$, that is, the CI is $\bar{X} + z^2_{\alpha/2}/(2n) \pm \sqrt{\bar{X} z^2_{\alpha/2}/n + z^4_{\alpha/2}/(4n^2)}$. If you thought that n would be very large, you can discard terms with higher powers of n in the denominator, giving the approximate large-sample interval $\bar{X} \pm z_{\alpha/2}\sqrt{\bar{X}/n}$. (Either answer will get full credit.)

(5), #38, p. 277. (a). This one is a prediction interval based on assumed normally distributed observations: a 95% interval is $.0635 \pm 1.96(.0065)\sqrt{26/25} = (.0505,\ .0765)$. Note that in the terminology of the problem, the width of the interval gives information about precision, and the confidence level is the 'reliability'.

(b). This one is a tolerance interval: since that is a topic I said we skipped. the 5 points for this part are extra credit. To give an interval so large that the probability is .95 that 95% or more of all pieces of laminate have warpage falling in the interval, use Table A.6 to obtain the interval $.0635 \pm (2.631)(.0065) = (.0464,\ .0806)$.

(6), #44, p. 280. The 95%interval for σ^2, directly from the book's formula using $n = 9$, is , is $(8(2.81^2)/17.535,\ 8(2.81^2)/2.180) = (3.603, 28.98)$, awfully wide. The CI for σ is obtained by taking square roots, and is $(1.898, 5.383)$.

(7), #52, p. 281. (a) With assumed-normal arsenic concentrations, the 95% CI is $24.3 \pm t_{4,.025} 4.1/\sqrt{5} = (19.21, 29.39)$. The interpretation must be stated carefully: retrospectively, after the observations we cannot make a probability statement about an unknown constant. But prospectively, before observing the data, if we intended to use this type of t-interval, we could say that the true unknown concentration would fall in the interval.

(b). The 90% upper confidence bound for σ is $(4(4.1)^2/.711)^{1/2} = 9.72$.

(c) The 95% prediction interval for the next water specimen is $24.3 \pm 2.776(4.1)\sqrt{6/5} = (11.83, 36.77)$.

(7), #12, pp. 294. (a). H_0 is that the average braking distance μ is ≥ 120. (b). Only $\bar{X}$ values which are too small indicates incompatibility of the data with H_0, so the answer is the region $R_2 = \{\mathbf{x} : \bar{x} \leq 115.2\}$.
(c). Significance level of R_2 is $\Phi((115.2 - 120)/(10/6)) = 0.002$.
(d). This is the power (probability of correct rejection) at $\mu = 115$, which

is $\Phi((115.2 - 115/(10/6)) = .548$. (e). Under H_0, the variable Z is $\mathcal{N}(0,1)$. With the region $\{z \leq -2.33\}$, the significance level is $\Phi(-2.33) = .01$, and that of the region $\{z \leq -2.88\}$ is $\Phi(-2.88) = .002$. (Note: $(115.2 - 120)/(10/6) = -2.88$, so this last region is R_2 !)