10.2 Limit theorems

We study next the asymptotic behaviour of $N(t)$ and its renewal function $m(t)$ for large values of t. There are four main results here, two for each of N and m. For the renewal process N itself there is a law of large numbers and a central limit theorem; these rely upon the relation (10.1.3), which links N to the partial sums of independent variables. The two results for m deal also with first- and second-order properties. The first asserts that $m(t)$ is approximately linear in t; the second asserts that the gradient of m is asymptotically constant. The proofs are given later in the section.

How does $N(t)$ behave when t is large? Let $\mu = \mathbb{E}(X_1)$ be the mean of a typical interarrival time. Henceforth we assume that $\mu < \infty$.

(1) Theorem. $$\frac{1}{t}N(t) \xrightarrow{\text{a.s.}} \frac{1}{\mu} \quad \textit{as } t \to \infty.$$

(2) Theorem. *If* $\sigma^2 = \text{var}(X_1)$ *satisfies* $0 < \sigma < \infty$, *then*

$$\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{\text{D}} N(0, 1) \quad \textit{as} \quad t \to \infty.$$

It is not quite so easy to find the asymptotic behaviour of the renewal function.

(3) Elementary renewal theorem. $\frac{1}{t}m(t) \to \frac{1}{\mu}$ *as* $t \to \infty$.

The second-order properties of m are hard to find, and we require a preliminary definition.

(4) Definition. Call a random variable X and its distribution F_X **arithmetic with span** λ (> 0) if X takes values in the set $\{m\lambda : m = 0, \pm 1, \dots\}$ with probability 1, and λ is maximal with this property.

If the interarrival times of N are arithmetic, with span λ say, then so is T_k for each k. In this case $m(t)$ may be discontinuous at values of t which are multiples of λ, and this affects the second-order properties of m.

(5) Renewal theorem. *If* X_1 *is not arithmetic then*

(6) $$m(t+h) - m(t) \to \frac{h}{\mu} \quad \textit{as} \quad t \to \infty \quad \textit{for all } h.$$

If X_1 *is arithmetic with span* λ, *then* (6) *holds whenever* h *is a multiple of* λ.

It is appropriate to make some remarks about these theorems before we set to their proofs. Theorems (1) and (2) are straightforward, and use the law of large numbers and the central limit theorem for partial sums of independent sequences. It is perhaps surprising that (3) is harder to demonstrate than (1) since it concerns only the mean value of $N(t)$; it has a suitably probabilistic proof which uses the method of truncation, a technique which proved useful in the proof of the strong law (7.5.1). On the other hand, the proof of (5) is difficult. The usual method of proof is largely an exercise in solving integral equations, and is not appropriate for inclusion here (see Feller 1971, p. 360). There is an alternative proof which is short, beautiful, and probabilistic, and uses 'coupling' arguments related to those in the proof of the ergodic theorem for discrete-time Markov chains. This method requires some results which appear later in this chapter, and so we defer a sketch of the argument until Example (10.4.21). In the case of arithmetic interarrival times, (5) is essentially the same as Theorem (5.2.24), a result about *integer-valued* random variables. There is an apparently more general form of (5) which is deducible from (5). It is called the 'key renewal theorem' because of its many applications.

In the rest of this chapter we shall commonly assume that the interarrival times are *not* arithmetic. Similar results often hold in the arithmetic case, but they are usually more complicated to state.

(7) Key renewal theorem. *If* $g : [0, \infty) \to [0, \infty)$ *is such that*:

(a) $g(t) \geq 0$ *for all* t,
(b) $\int_0^\infty g(t)\,dt < \infty$,
(c) g *is a non-increasing function,*

then

$$\int_0^t g(t-x)\,dm(x) \to \frac{1}{\mu}\int_0^\infty g(x)\,dx \quad \text{as} \quad t \to \infty$$

whenever X_1 *is not arithmetic.*

In order to deduce this theorem from the renewal theorem (5), first prove it for indicator functions of intervals, then for step functions, and finally for limits of increasing sequences of step functions. We omit the details.

Proof of (1). This is easy. Just note that

$$\textbf{(8)} \qquad T_{N(t)} \leq t < T_{N(t)+1} \quad \text{for all } t.$$

Therefore, if $N(t) > 0$,

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1}\left(1 + \frac{1}{N(t)}\right).$$

As $t \to \infty$, $N(t) \xrightarrow{\text{a.s.}} \infty$, and the strong law of large numbers gives

$$\mu \leq \lim_{t\to\infty}\left(\frac{t}{N(t)}\right) \leq \mu \quad \text{almost surely.}$$

■

Proof of (2). This is Problem (10.5.3). ■

In preparation for the proof of (3), we recall an important definition. Let M be a random variable taking values in the set $\{1, 2, \dots\}$. We call the random variable M a *stopping time* with respect to the sequence X_i of interarrival times if, for all $m \geq 1$, the event $\{M \leq m\}$ belongs to the σ-field of events generated by $X_1, X_2, \dots, X_m$. Note that $M = N(t) + 1$ is a stopping time for the X_i, since

$$\{M \leq m\} = \{N(t) \leq m - 1\} = \left\{\sum_{i=1}^{m} X_i > t\right\},$$