February 10, 2020

Solution to Selected STAT 650 HW1 Problems

Lefebvre #12. By inspection of P, we have $P_2(\text{hit 0 before 3}) = 0$. From this it is easy to check that

$$x = P_1(\text{hit 0 before 3}) = 1/2 + (1/4)P_1(\text{hit 0 before 3}) + (1/4)P_2(\text{hit 0 before 3})$$
$$= 1/2 + (1/4)x = 2/3$$

Lefebvre #15. The tricky aspect of this problem is to understand the (only) sense in which the limit could exist. A further point is that initially there is no Markov Chain in the problem. If we let $Y_k = I[\text{success in the k'th trial}]$, then $X_n = Y_{n-1} + Y_{n-2}$. But Y_n is not a Markov chain because its transition probabilities depend on 2 steps of memory and not just 1. Thus, if $Z_n = (Y_n, Y_{n-1}) \in \{0,1\}^2$, then we verify that Z_n **is** a Markov Chain with transition probabilities

$$P(Z_{n+1} = (a,b) \,|\, Z_n = (i,j)) = P(Y_{n+1} = a \,|\, Y_n = b,\, Y_{n-1} = j)\, I_{[i=b]}$$
$$= (b + j + 1)/(b + j + 2)\, I_{[i=b]}$$

Now consider whether $p_n = (Y_{n-1} + Y_{n-2} + 1)/(Y_{n-1} + Y_{n-2} + 2)$ could converge with probability 1 or in probability. For this to hold, there would have to be some random variable W such that

$$\text{for all} \quad \epsilon > 0, \qquad P(|Y_{n-1} + Y_{n-2} - W| > \epsilon) \to 0 \quad \text{as} \quad n \to \infty$$

Clearly W would have to be integer-valued, but this limiting condition is impossible because (since the p_k values are always bounded between 1/2 and 3/4), all configurations for $(Y_{n-1}, Y_{n-2}) \in \{0,1\}^2$ occur infinitely often. In other words, in spite of the way the problem is stated, the limit of p_n **cannot** exist in probability (and therefore also not with probability 1).

But as we learn in further work with Markov Chains, it makes sense to ask whether the random variables p_n converge in distribution, and there the answer is yes. If we regard the Markov chain Z_n above, and look at its transition matrix (with states written in the order $(0,0), (1,0), (0,1), (1,1)$)

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}$$

There is a unique **invariant** or **stationary** probability for this irredicible Markov Chain, given by $\pi = (2,3,3,8)/16$, which gives the respective limiting probabilities that Z_n is respectively equal to $(0,0),(1,0),(0,1),(1,1)$. And it follows from this that the limiting ($n \to \infty$) probability distribution for $Y_{n-1}+Y_{n-2}$ has probability mass function $(1/8)\delta_0 + (3/8)\delta_1 + (1/2)\delta_2$, which means that the limiting probability distribution for p_n has probability mass function $(1/8)\delta_{1/2} + (3/8)\delta_{2/3} + (1/2)\delta_{3/4}$.

Serfling #17. It is straightforward to calculate that

$$P(X_{n+1}=j \,|\, X_n=i,\ X_t=i_t \ \text{ for } \ t<n) \ = \ \sum_{k=1}^{i} I_{[j=i]} \ + p_j \, I_{[j>i]}$$

Then the sequence τ_n remains finite with probability 1 (which is necessary in order that X_n' be well-defined) only if there are infinitely many positive values p_k, in which case

$$P(X'_{n+1}=j \,|\, X'_n=i,\ X'_t=i_t \ \text{ for } \ t<n) \ = \ I_{[j>i]}\, p_j \, I_{[j>i]} / \sum_{k=i+1}^{\infty} p_k$$

If there is a largest value $k=k_*$ for which $p_k>0$ then X_n' is not well-defined for $n>1$, and the Markov chain X_n has all states other than k_* transient, with k_* absorbing (i.e., a singleton recurrent class). If there is no such largest k, then all states for both chains are transient.

Extra Problem (I) Here the solution steps are to find two conditional densities and put them together into an unusual kind of "mixed-type" joint probability distribution, as follows. Let $p = P(X<Y) = 0.5 \int_0^2 e^{-x}dx = (1-e^{-2})/2$, and calculate

$$f_{X,Y\,|\,X<Y}(x,y) \ = \ \frac{1}{2p}\, I_{[0<x<\min(2,y)]}\, e^{-y} \ , \ \ f_{Y\,|\,Y\le X}(y) \ = \ \frac{I[0<y<2]}{2(1-p)}\,(2-y)e^{-y}$$

leading to the joint probability distribution that can be written in the form (for positive infinitesimal dz, dw)

$$P(\min(X,Y)\in[z,z+dz),\ Y-\min(X,Y)\in[w,w+dw)) \ =$$

$$I_{[z\in(0,2)]}\, dz \left(I_{[w>0]}\, p\, f_{X,Y\,|\,X<Y}(z,z+w)\, dw \ + \ I_{[w=0]}\,(1-p)\, f_{Y\,|\,Y\le X}(z) \right)$$