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(Partial) Solutions to STAT 650 HW2 Problems

Serfozo #25. Fix $i, j \in C \subset S$ with respective periods d_i, d_j, where C is an irreducible class. The problem tells us to define m, n as integers (the book says the smallest ones, but that does not matter) such that p^n_{ij}, $p^m_{ji} > 0$, and let $a = p^n_{ij}\, p^m_{ji}$, which is also positive. Then for all $t \geq 0$, the multistep probability $p^{t+n+m}_{ii} \geq p^n_{ij}\, p^t_{jj}\, p^m_{ji} = a\, p^t_{jj}$, and similarly $p^{t+m+n}_{jj} \geq a\, p^t_{ii}$. By definition of *period*, any $t \geq 0$ with $p_{jj} > 0$ is divisible by d_j, and any $t \geq 0$ with $p_{ii} > 0$ is divisible by d_i, and $m+n$ is divisible by both d_i and d_j. It follows that for any t with $p^t_{jj} > 0$, $t+m+n$ and therefore also t is divisible by d_i, so that $d_j \geq d_i$ (since d_j is the gcd. Similarly, for any t with $p^t_{ii} > 0$, $t+m+n$ and therefore t is divisible by d_j, so that $d_j \leq d_i$. Therefore $d_i = d_j$.

Serfozo #36. With reference to Example 71 from p.45 of the book, the meaning seems to be that each single 'time step' n indexes a single job processed, but that a job of type i might immediately follow another job of type i. In that case, presumably $v(i,i) = 0$ for all i. The intended method of calculating γ given as a probability-1 or in-probability limit of average costs is to check that if the chain is ergodic with stationary distribution π and starts in i_0, then

$$E\Big(\frac{1}{n}\sum_{t=1}^{n} v(X_{t-1}, X_t)\Big) \;=\; \frac{1}{n}\sum_{t=1}^{n}\sum_{i,j\in S} p^t_{i_0,i}\, p_{ij}\, v(i,j) \;\to\; \sum_{i,j\in S} \pi_i\, p_{ij}\, v(i,j)$$

since $p^t_{i_0,i} \to \pi_i$ as $t \to \infty$. The corresponding limiting per-job cost of switches from i is $\gamma(i) = \pi_i \sum_{j\in S} p_{i,j}\, v(i,j)$. To make the result of this problem rigorous, we will discuss in future classes the convergence with probability 1 of the averages $n^{-1}\sum_{t=1}^{n} v(X_{t-1}, X_t)$ as $n \to \infty$, and some additional requirement related to the uniform boundedness of the expectation of $v(X_{t-1}, X_t)$. For example if $\sum_{j\in S} p_{ij} v_{i,j} \leq C$ for some finite constant C not depending on i, then the limit w.p.1 and in expectation can be made rigorous.

Serfozo #37. The renewal age process of Example 20 was given in terms of partial sums $T_k = \sum_{a=1}^{k} \xi_a$ of *iid* discrete random variables ξ_a with distribution function F, by $X_n = n - \max\{T_{k-1} :\; k \geq 1,\, T_{k-1} \leq n\}$. The resulting transition probabilities (also given in Example 20) are as follows. If $X_n = 0$, then $X_n = T_{k-1}$ for some $k \geq 1$, and $X_{n+1} = I_{[\xi_k > 1]}$ which means $p_{0,1} = 1 - p_{0,0} = 1 - F(1)$. If $X_n = i > 0$, then for some $k \geq 1$ $T_{k-1} = n - i$ and $\xi_k > i$, with $X_{n+1} = (i+1)\, I_{[\xi_k > i+1]}$, and $p_{i,i+1} = 1 - p_{i,0} = P(\xi_k > i+1 \,|\, \xi_k > i) \;=\; (1 - F(i+1))/(1 - F(i))$.

In this problem, if there is a finite maximum lifetime for the renewing devices, i.e., if there is a finite m_* such that $F(m_*) = 1$, then the irreducible closed class of states is $S = \{0, 1, \ldots, m_* - 1\}$, and otherwise the state-space $S = \{0, 1, \ldots\}$ is irreducible. Since $p_{0,0} > 0$, the chain is aperiodic. To verify ergodicity, it suffices to check that the probability vector π stated in the problem is actually a stationary distribution.

Serfozo #44. The problem does not define "state-space" and "location" precisely. If "location" is regarded as the edge most recently traversed, and this is synonymous with "state", then $|S|$ is the number of edges, and the problem is correct as stated. If instead you want to define the state space $S = G$ as the set of vertices or nodes of the graph, then the formula for stationary distribution is not correct because the sum of π_i entries does not sum to 1. Indeed, as part of this problem you must verify the (easy) assertion that $\sum_{i \in G} |G_i|$ is exactly twice the number of edges in the graph.

Lefebvre #20. The "ladder chain" described here is irreducible by inspection. The key in this problem is to realize that starting from state 0, the stopping-time τ_0 has probability mass function given for $k \geq 1$ by

$$P_0(\tau_0 = k) \;=\; \alpha_0 \,\cdot\, \alpha_1 \,\cdots\, \alpha_{k-1}\,(1 - \alpha_k)$$

where the product $\alpha_0 \cdots \alpha_{k-1}$ is taken to be 1 when $k = 0$. This equation implies that $P_0(\tau_0 \geq k) = \prod_{t=0}^{k-1} \alpha_t$. The condition $P_0(\tau_0 < \infty)$ for recurrence is simply that $P_0(\tau_0 \geq k) \to 0$ as $k \to \infty$, and the criterion for positive-recurrence is that $\sum_{k=1}^{\infty} k \cdot P_0(\tau_0 = k) = \sum_{k=1}^{\infty} P_0(\tau_0 \geq k) < \infty$.