March 13, 2020

Partial Solutions to STAT 650 HW3 Problems

Serfozo #27. We discussed essentially this problem (with the two Markov-chain components sharing the same transition matrix) in connectiuon with the coupling proof of convergence of multistep transition probabilities for aperiodic irreducible positively recurrent chains. The only tricky part (for which I referred you to Lemma 4.2 of Durrett) was to establish irreducibility of the two-component chain $Z_n = (X_n,\, X'_n)$.

Serfozo #39. Let $Y_n \sim \text{Poisson}(\lambda)$ be the number of arrivals at time n (i.e., between n and $n+1$), independent of $X_1, \ldots, X_n$ and Z_n, where $Z_n \sim \text{Binom}(X_n, p)$ given X_n is the number of visitors at the site at time n who leave before $n+1$. Then

$$P(X_{n+1} = j \,|\, X_n = i, X_{n-1}, \ldots, X_0) \;=\; e^{-\lambda} \sum_{y=0}^{j} \frac{\lambda^y}{y!} \binom{i}{i+y-j} p^{i+y-j} \, (1-p)^{j-y}$$

So X_n is a homogeneous Markov Chain, obvously irreducible and aperiodic. Note that for X_n to be ergodic, the distribution of $X_{n+1} = X_n - Z_n + Y_n$ and X_n would both be π, and it is clear that the expectation of $Y_n - Z_n$ must be 0, so that $\lambda = p\, E_\pi(X_n)$, and $\sum_{k \ge 0} k\, \pi_k = \lambda/p$. So if π were a Poisson distribution, its parameter must be λ/p.

As we will learn in connection with Poisson processes: if $X \sim \text{Poisson}(\beta)$ and given X, $W \sim \text{Binom}(X, q)$, then $W \sim \text{Poisson}(\beta q)$. You can verify this easily using moment generating functions. Then $X_n \sim \text{Poisson}(\beta)$ implies (with $q = 1-p$) that $X_n - Z_n \sim \text{Poisson}((1-p)\beta)$ and $X_n - Z_n + Y_n \sim \text{Poisson}((1-p)\beta + \lambda)$. It is easy to see that if $\beta = \lambda/p$, then $X_{n+1} \sim \text{Poisson}(\beta)$, verifying stationarity of π equal to a Poisson(λ/p) distribution, assuming no condition other than $\lambda, p > 0$.

Serfozo #50. This M/M/1 chain is a Birth-Death chain with

$$\begin{aligned} p_{i,j} = \quad & p\, I_{[i=0,\, j=1]} \;+\; (1-p)\, I_{[i=0, j=0]} \;+\; p(1-q)\, I_{[i>0, j=i+1]} \\ & + q(1-p)\, I_{[i>0, j=i-1]} \;+\; (pq + (1-p)(1-q))\, I_{[i=j>0]} \end{aligned}$$

Irreducibility and aperiodicity are obvious, and we showed reversibility whenever the chain is positive-recurrent, which in turn is confirmed when there is an invariant distribution. The stationary distribution given in the problem statement, whenever $\rho = p(1-q)/(q(1-p)) < 1$ (i.e., whenever $p < q$), is the one determined in class by the detailed-balance (reversibility) equations, with $\pi_k = \pi_1\, \rho^{k-1}$ for $k \ge 1$, and $\pi_1 = \rho \pi_0 / (1-q)$.

The cost-structure in the problem can be expressed as a cost assessed at each discrete time $k+1$ for the state of the system between k and $k+1$, equal to $h \cdot X_k + s\, I_{[X_k \geq 1]}$. The average long-term cost per unit time is the same as the expected cost on each unit-time interval under the stationary distribution, or

$$(1-\pi_0)\, s \;+\; h \sum_{m \geq 0} m\, \pi_m \;=\; ps/q \;+\; h(1-\rho)(p/q) \sum_{m=1}^{\infty} m\rho^{m-1}$$

which is equal to $ps/q + p(1-p)h/(q-p)$ and differs in its first term from the answer given in the problem statement. If there is also a reward R for every customer served, then the total reward per step in a large number n of steps is approximately R times the number of times at which a new customer arrives up to n (since arrivals and service-completions balance out asymptotically), or

$$R \cdot (n\, p)/n \;=\; Rp$$

To check this answer for the reward term, let's calculate it another way, by counting service-completions, which occur with probability q at each time-point k when there is at least one person in line. Therefore the long-term reward per unit time is

$$R\,(1-\pi_0) nq/n \;=\; R\,(p/q)\, q \;=\; Rp$$

Lefebvre #38. The long-term proportion of steps that the chain spends in state j is the j'th entry of an invariant probability distribution, which exists and is unique even though the chain has period 2. Solving the invariance equations gives the solution: $\pi = (1,2,2,1)/6$. For part (c), let $v_j = P_j(T_0 < T_3)$ for $j = 1,2$. Then the first-step equations are

$$v_1 \;=\; 0.5(1+v_2)\,, \quad v_2 = 0.5 v_1 \qquad \Longrightarrow \qquad v_1 = 2/3$$

Lefebvre #41. The transition mechanism seems to be that given X_n, the expected proportion p of A's in the population after a single randomly chosen individual undergoes the 'mutation' step is

$$X_n/N \;+\; (X_n/N)\,(-1+1-\alpha) \;+\; (1-X_n/N)\,\beta \;=\; \beta + (X_n/N)\,\cdot\,(1-\beta-\alpha)$$

and then $X_{n+1} \sim \text{Binom}(N,p)$ given X_n and $p = \beta + (1-\beta-\alpha)X_n/N$. This implies

$$p_{i,j} \;=\; \binom{N}{j} \left(\beta + (1-\beta-\alpha)i/N\right)^j \left(1-\beta-(1-\beta-\alpha)i/N\right)^{N-j}$$

When $\alpha = 0,\ \beta \in (0,1)$, it is easy to see from this last equation that the state N is absorbing (and so recurrent), and all other states communicate and are transient.

Extra (A). As more than one student mentioned to me, a very reasonable guess in this problem before doing any work is that the criterion for reversibility is symmetry of the 3×3 transition matrix P, i.e. $a = b \in [0, 2/3]$, and this is the correct answer for the problem. To prove it, note that when $a = b$ and $p_{0,0} = p_{2,2} = 2/3 - a$, then the matrix P is doubly stochastic (all rows and columns sum to 1, which implies that the unique invariant distribution is $\pi_j = 1/3$ for $j = 0, 1, 2$), and the symmetry of P then exactly expresses the reversibility condition. Conversely, if the chain with general P (and $p_{0,0} = 2/3 - a$, $p_{2,2} = 2/3 - b$) is reversible, then the reversibility condition implies $p_{0,1}/p_{1,0} = 1 = \pi_1/\pi_0$ and similarly $p_{1,2}/p_{2,1} = 1 = \pi_2/\pi_1$, so that $\pi_j = 1/3$ for all $j = 0, 1, 2$. Then $1/3 = \pi \cdot (2/3 - a, 1/3, b) = (1 - a + b)/3$ implies $a = b$.

Extra (B). It is easy to see that the chain is irreducible, and letting $F = \{0, 1, \ldots, 9\}$,

$$P_k\big(\tau_F > n \,\big|\, \tau_F > n-1\big) \;\le\; 1 \;-\; 1/\sqrt{k + 20(n-1)} \qquad \text{for all} \quad k \ge 10, \;\; n \ge 1$$

Therefore we complete the proof of (a) by estimating for $k \ge 10$,

$$P_k(\tau_F > n) \;\le\; \prod_{r=0}^{n-1} \big(1 - 1/\sqrt{k + 20r}\big) \;\le\; \exp\Big(-\sum_{r=0}^{n-1} (k + 20r)^{-1/2}\Big) \;\to\; 0 \qquad \text{as} \quad n \to \infty$$

Then part (b) follows easily by (a) together with the remark (by recurrence of finite irreducible chains) that $P_i(\text{hit } \{0, 10, 11, 12, \ldots\}) = 1$.

Extra (C). (a) This branching process has mean number of offspring $\mu = 1.2$, so the extinction probability is the smallest positive root of $s = 0.1 + 0.6s + 0.3s^2$, or $s_1 = 1/3$. (b) $E(X_4) = (1.2)^4$, and $\mathrm{Var}(X_1) = 0.36$, while for each $k \ge 0$,

$$\mathrm{Var}(X_{k+1}) = E(\mathrm{Var}(X_{k+1} \,|\, X_k)) + \mathrm{Var}(E(X_{k+1} \,|\, X_k)) \;=\; (0.36)\,(1.2)^k + 1.44\,\mathrm{Var}(X_k)$$

(c) $E(X_4 \,|\, X_3 > 1) \;=\; E(X_4 \;-\; 1.2 X_3\, I_{[X_3 = 1]})/(1 - P(X_3 \le 1))$ and with $\phi(s) = 0.1 + 0.6s + 0.3s^2$, the constant-term and coefficient of s in $\phi \circ \phi \circ \phi(s)$ are respectively $P(X_3 = 0)$ and $P(X_3 = 1)$, and these are respectively $0.20577, \; 0.27633$.