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Handout on Unique Solution of Kolmogoroff DE's

This handout is intended to clear up some question about the existence and unique determination of continuous-time HMC's with specified transition intensity matrices Q. We have seen in class that regular jump HMC's always have transition probability matrices that satisfy the Backward Kolmogoroff differential equation $P'(t) = QP(t)$, and the Brémaud book showed that when

$$\lambda \equiv \sup_{i \in S} q_i < \infty \tag{1}$$

the forward equations are also satisfied. Let us take those results as given, along with the book's later justification for all regular jump HMC's the properties of stability (non-explosiveness, which says $\sup_n \tau_n = \infty$ with probability 1, where τ_n is the time of the HMC's n'th state-transition) and conservativeness (i.e., $\sum_{j \in S:, j \neq i} q_{ij} = q_i$ for all $i \in S$).

The primary remaining question is how to connect the analytical object of our study, the transition probability semigroup $P(t)$, with the probabilistic embedded-chain and waiting-time picture we have discussed in class. Both of those constructions lead to local intensities, small-time limits of difference quotients of transition probabilities, summarized in the Q matrix. To know that they lead to exactly the same description, we must establish that the transition probabilities $P_{ij}(t)$ for all times t are uniquely determined by the Kolmogoroff differential equations.

It is actually very easy to prove uniqueness of solutions of the forward and backward equations under the condition (1). The method is a lot like the 'Gronwall's Inequality' proofs you have probably seen in undergraduate-level treatments of the uniqness of ODE solutions. The main additional wrinkle here is that the Kolmogoroff forward and backward equations are each systems of infinitely many equations if S is infinite.

As a preliminary step, check by induction under (1) that $\sup_i \sum_j |(Q^n)_{ij}| \leq (2\lambda)^n$ for all $n \geq 1$, and therefore that $e^{Qt} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n$ converges absolutely uniformly on bounded intervals of t, and is thus differentiable and a solution of both the forward and backward equations.

Consider the backward equations first, and suppose that $P(t)$ and $L(t)$ are both solutions of the system $P'(t) = QP(t)$, $L'(t) = QL(t)$, with $L(0) = P(0) = I$ and $\sum_{j \in S} P_{ij}(t)$, $\sum_{j \in S} L_{ij}(t)$ bounded uniformly in i for sufficiently small t. Recall that the backward equations

$$P'_{ij}(t) = -q_i \, P_{ij}(t) + \sum_{k \in S: \, k \neq i} q_{ik} \, P_{kj}(t)$$

along with their initial conditions $P(0) = I$ imply that

$$e^{q_i t} P_{ij}(t) - \delta_{ij} \;=\; \int_0^t \frac{d}{ds}\left(e^{q_i s} P_{ij}(s)\right) ds \;=\; \int_0^t e^{q_i s} \sum_{k\in S:\, k\neq i} q_{ik}\, P_{kj}(s)\, ds \tag{2}$$

and similarly for $L_{ij}(t) - \delta_{ij}$ in terms of $L_{ik}(t)$ with $k \neq i$. Note that the right-hand summation converges and is integrable because of the conservative property of the intensities. It follows from (2) that $m(t) \equiv \sup_i m_i(t)$ where $m_i(t) \equiv \sum_{j\in S} P_{ij}(t)$ satisfy

$$e^{q_i t}\, m_i(t) \le 1 + \int_0^t \lambda\, m(s) ds$$

and similarly for analogous quantities with P replaced by L, and

$$K(t) \;=\; \sup_i \Big| \sum_{j\in S} \left(P_{ij}(t) - L_{ij}(t)\right) \Big|$$

is continuous and satisfies $K(0) = 0$ and

$$K(t) \le \sup_i \int_0^t e^{q_i(s-t)} \sum_{k:k\neq i} q_{ik} K(s)\, ds \;\le\; \lambda\, K(s)\, ds$$

Thus $K(t)$ is identically 0, from which we conclude for all i that $\sum_{j\in S} P_{ij}(t) \equiv \sum_{j\in S} L_{ij}(t) \equiv 1$. Now fix an arbitrary $i \in S$, and let

$$r(t) \;\equiv\; \sum_{j\in S} |P_{ij}(t) - L_{ij}(t)| \tag{3}$$

It follows by subtracting the integral equations (2) for $P_{ij}(t)$ and $L_{ij}(t)$ that

$$r(t) \;\le\; \sum_{j\in S} \Big| \int_0^t e^{q_i(s-t)} \sum_{k\in S:\, k\neq i} q_{ik}\left(P_{kj}(s) - L_{kj}(s)\right) ds \Big| \;\le$$

$$\le \int_0^t e^{q_i(s-t)} \sum_{k:\, k\neq i} q_{ik}\, r(s)\, ds \;\le\; \lambda \int_0^t r(s)\, ds$$

which again implies that $r(t)$ is identically 0. This proves the uniquess of the solution $\exp(tQ)$ for the backward equation.

A similar idea works for the forward equation. Now the integral equation becomes

$$e^{q_j t} P_{ij}(t) \;-\delta_{ij} \;=\; \int_0^t e^{q_j s} \sum_{k:\, k\neq j} P_{ik}(s)\, q_{kj}\, ds \tag{4}$$

Arguing exactly as before in terms of a function like $K(t)$ above, we find that for any two systems of solutions $P_{ij}(t), L_{ij}(t)$ of (4), for each of which the quantities $\sup_i \sum_j P_{ij}(t)$, $\sup_i \sum_j L_{ij}(t)$ are bounded for small t, for all t, $\sum_j P_{ij}(t) \equiv \sum_j L_{ij}(t) \equiv 1$. Again using $r(t)$ as defined in (3) for arbitrary fixed i, we now reason using (4) that

$$r(t) \;\le\; \sum_{j\in S} \Big| \int_0^t e^{q_j(s-t)} \sum_{k\in S:\, k\neq j} q_{kj}\,(P_{ik}(s) - L_{ik}(s))\,ds \Big| \;\le$$

$$\le \int_0^t \sum_j e^{q_j(s-t)} \sum_{k:\, k\neq j} |P_{ik}(s) - L_{ik}(s)|\, q_{kj}\,ds$$

$$\le \int_0^t \sum_k |P_{ik}(s) - L_{ik}(s)| \sum_{j:\, j\neq k} q_{kj}\,ds \;\le\; \lambda \int_0^t r(s)\,ds$$

which again implies $r(t) \equiv 0$ because $r(0) = 0$.