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Two Loose Ends from Stat 650 Class

Here are two comments & calculations to clear up things I showed in class that may have caused confusion.

(1). Condition for stationary distribution in Birth-Death Chains.

Recall that we showed in class that the Birth-Death chains on $S = \{0, 1, 2, \ldots\}$ with $P_{k,k+1} = \mu_k > 0,\ P_{k,k-1} = \lambda_k > 0$ for $k \ge 1$ and $P_{0,1} = 1$, are

$$\text{irreducible if and only if} \qquad \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{\lambda_j}{\mu_j} \;=\; \infty$$

We showed also that the condition $\underline{\pi}^{\text{tr}} P \;=\; \underline{\pi}^{\text{tr}}$ for an infinite vector $\underline{\pi} = (\pi_0, \pi_1, \ldots)$ implies

$$\nu_{k+1} - \nu_k \;=\; (\nu_k - \nu_{k-1}) \frac{\lambda_k}{\mu_k} \;, \quad k \ge 1, \quad \text{for} \quad \nu_k \;\equiv\; \Big\{ \prod_{j=0}^{k-1} \frac{\lambda_{j+1}}{\mu_j} \Big\} \pi_k \tag{1}$$

It followed by induction from this equation that for $k \ge 1$

$$\nu_{k+1} - \nu_k \;=\; \Big[\prod_{j=1}^{k} \frac{\lambda_j}{\mu_j} \Big] (\nu_1 - \nu_0) \quad \Longrightarrow \quad \nu_{k+1} \;=\; \nu_1 + \sum_{j=1}^{k} \Big[\prod_{i=1}^{j} \frac{\lambda_i}{\mu_i} \Big] (\nu_1 - \nu_0) \tag{2}$$

In order for $\underline{\pi}$ to be a stationary distribution, the quantities ν_k defined inductively from an initial pair $(\nu_0,\, \nu_1)$ must be nonnegative, not all 0, and such that the numbers π_k related to ν_k through (1) are summable. There were two cases. First, suppose $\nu_1 = \nu_0$ is nonzero, necessarily positive. Then it follows that $\nu_k = \nu_0$ for all k, and

$$\pi_k \quad \text{is summable if and only if} \qquad \sum_{k=1}^{\infty} \Big[\prod_{j=0}^{k-1} \frac{\mu_j}{\lambda_{j+1}} \Big] \;<\; \infty \tag{3}$$

We wanted to argue that this is actually the **only** condition for existence of a stationary probability distribution (obtained by dividing $\underline{\pi}$ through by $\sum_{j=0}^{\infty} \pi_j$). For this, we needed to exclude, **assuming recurrence**, the possibility that a solution ν_k of (2) with $\nu_1 \ge 0,\ \nu_1 \ne \nu_0$ could lead to nonnegative summable π_k.

If $\nu_1 \neq \nu_0$ and ν_{k+1} is given by the last part of (2), then the irreducibility condition implies that $\nu_{k+1} \geq 0$ for large k only if $\nu_1 > \nu_0$. With $\nu_1 - \nu_0 > 0$, summability of π_k from (1)-(2) would imply

$$\sum_{k=0}^{\infty} \Big[\prod_{j=0}^{k-1} \frac{\mu_j}{\lambda_{j+1}}\Big]\Big(\sum_{m=1}^{k} \Big[\prod_{i=1}^{m} \frac{\lambda_i}{\mu_i}\Big]\Big) \; < \; \infty$$

and the summation is only decreased if we restrict the inner sum over m to the single value $m = k$. Thus the summbility condition of the last equation implies

$$\sum_{k=0}^{\infty} \Big[\prod_{j=0}^{k-1} \frac{\mu_j}{\lambda_{j+1}}\Big] \Big[\prod_{i=1}^{k} \frac{\lambda_i}{\mu_i}\Big] \; < \; \infty \quad \Longrightarrow \quad \sum_{k=0}^{\infty} \frac{\mu_0}{\mu_k} \; < \; \infty$$

But this last condition is impossible because $\mu_k \leq 1$ for all k. Thus the only way to achieve summable π_k from (1) is to have $\nu_1 = \nu_0$, and for recurrent Birth-Death chains, existence of a stationary probability distribution is equivalent to condition (3).

(2). 'Cycle Trick' in Durrett, p. 84.

We considered the irreducible HMC $\{X_t\}$ on countably infinite state-space S, and assumed $x \in S$ had the property $E_x(T_x) < \infty$, which implies that x is recurrent (since the expectation could not be finite without $P_x(T_x < \infty) = 1$), and therefore all other states are too.

With fixed x, the key definitions were

$$\mu(y) \; = \; E_x\Big(\sum_{t=1}^{T_x} I_{[X_t=y]}\Big) \; = \; E_x\Big(\sum_{t=0}^{T_x-1} I_{[X_t=y]}\Big) \; = \; E_x\Big(\sum_{t=0}^{\infty} I_{[T_x>t]} \, I_{[X_t=y]}\Big)$$

and changing as in Durrett without the $\bar{p}_t(x,y)$ notation,

$$\mu(y) \; = \; \sum_{t=0}^{\infty} P_x(X_t = y, \; T_x > t) \tag{4}$$

Then as in Durrett, we calculate first for $k \neq x$,

$$\sum_{y \in S} \mu(y) \, P_{y,k} \; = \; \sum_{t=0}^{\infty} \sum_{y \in S} P_x(X_t = y, \; X_{t+1} = k, \; T_x > t)$$

$$= \sum_{t=0}^{\infty} P_x(X_{t+1} = k,\ T_x > t+1) \ = \ \sum_{s=0}^{\infty} P_x(X_s = k,\ T_x > s) \ = \ \mu(k)$$

and then, for $k = x$, we know by the definition that $\mu(x) = 1$, and

$$\sum_{y \in S} \mu(y)\, P_{y,x} \ = \ \sum_{t=0}^{\infty} \sum_{y \in S} P_x(X_t = y,\ X_{t+1} = x,\ T_x > t) \ = \ \sum_{t=0}^{\infty} P(T_x = t+1) \ = \ 1$$

Thus we have shown for all $k \in S$, that $\sum_{y \in S} \mu(y)\, P_{y,k} \ = \ \mu(k)$. Note also from equation (4) that

$$\sum_{y \in S} \mu(y) \ = \ \sum_{t=0}^{\infty} \sum_{y \in S} P_x(X_t = y,\ T_x > t) \ = \ \sum_{t=0}^{\infty} P_x(T_x > t) \ = \ E_x(T_x)$$

Thus $(\mu(y)/E_x(T_x) : y \in S)$ is a stationary probability vector.

This is our starting point for next class.