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Sample Problems for In-Class Stat 650 Test

(1). Suppose that the transition matrix $\mathbf{P}$ of a homogeneous Markov chain on the state-space $S = \{0, 1, 2, \ldots\}$ has the following three properties (for all $k \geq 0$) : (i) $p_{k0} \geq 1/(2k+2)$, (ii) $p_{k,k+1} > 0$, and (iii) $p_{k,j} = 0$ whenever $j > k + 10$. Then use standard results about Markov chains to show that every state is recurrent.

Hint: *use the conditions to obtain an upper bound on*

$$P(X_{k+1} \neq 0 \,|\, X_1,\, X_2,\, \ldots,\, X_k \neq 0)$$

(2). Consider the homogeneous Markov chain with states 1, 2, 3, 4 and one-step transition matrix

$$P \;=\; \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.2 & 0 & 0.4 & 0.4 \end{pmatrix}$$

(a) Find $E(\inf\{n \geq 1 : \; X_n = 3\} \,|\, X_0 = 3)$.

(b) Find $P(X_n$ hits 3 before $2 \,|\, X_0 = 1)$. The event $[X_n$ hits 3 before 2$]$ is defined formally as the event $\Big[\inf\{n \geq 1 : \; X_n = 3\} < \inf\{n \geq 1 : \; X_n = 2\}\Big]$.

(3). Using any formulas or theorems from the book or from class, prove the recurrence and null-recurrence of the homogeneous discrete-state Markov chain with state-space equal to the integers and with one-step transition probabilities

$$p_{jk} \;=\; \frac{1}{3} \cdot \Big\{ I_{[|k-j| \leq 1]} \Big\}$$

(4). Suppose that a HMC on a countable state-space S has a transition matrix P such that, for some probability vector $\mathbf{v}$ with all positive entries v_i,

$$v_i \, P_{ij} \;=\; v_j \, P_{ji}$$

Show that this HMC is recurrent.

Hint: *calculate the expected number of visits to a fixed state i in the first n transition steps when the chain is started with the distribution* $\mathbf{v}$*, i.e. when* $P(X_0 = j) = v_j$ *for all* j*.*

(5). For the HMC with states $S = \{1,2,3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

(a) Find $\lim_{n\to\infty} P^n_{31}$.

(b) For a chain with this transition matrix, find the long-term proportion of visits to state 3 which occur immediately after a visit to 2. That is, find

$$\lim_{N\to\infty} \frac{\#\{n = 1,\ldots,N : \ X_{n-1} = 2,\ X_n = 3\}}{\#\{n = 1,\ldots,N : \ X_n = 3\}}$$

(6). For the HMC with states $1,\ldots,5$ and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

find $\lim_{n\to\infty} P^n_{1k}$ for all $k = 1,\ldots,5$.

(7). Explain (with brief justification or counterexample) whether each of the following statements is true or false.

(i) For each finite-state aperiodic irreducible HMC, there is a finite power of the transition matrix all entries of which are positive.

(ii) Every discrete-state birth-death HMC is reversible.

(iii) The limit of P^n for every irreducible finite-state chain exists.

(iv) For an irreducible aperiodic HMC with countable states, the invariant distribution is unique whenever one exists.

(v) If P and Q are irreducible transition matrices on the same state space and $P_{i0} \ge Q_{i0}$ for all i, with $P_{ij}/(1-P_{i0}) = Q_{ij}/(1-Q_{i0})$ whenever $i,j \ne 0$, then recurrence of the chain with transition Q implies recurrence for the chain with transition P.