Stat 650 February 5, 2007

Non-Markovian Example

As indicated in class, this is an exampled of a *lumped-state* random sequence constructed from a Homogeneous Markov Chain, and we supply calculations to show the lumped-state chain is non-Markovian.

Let $\{X_k : \; k \geq 0\}$ be a homogeneous Markov Chain with state-space $S = \{0, 1, 2\}$ defined by $X_0 = 1$ and transition probability matrix

$$\mathbf{P} \;=\; \Big(P(X_{k+1} = j \,|\, X_k = i)\Big)_{i,j=0,1,2} \;=\; \begin{pmatrix} .8 & .1 & .1 \\ .2 & .7 & .1 \\ 0 & .1 & .9 \end{pmatrix}$$

Define a binary-valued random sequence

$$Y_k \;=\; I_{[X_k \geq 1]} \;=\; \begin{cases} 1 & \text{if} \quad X_k = 1, 2 \\ 0 & \text{if} \quad X_k = 0 \end{cases}$$

Then one easily calculates $P(Y_1 = 1) = P(X_1 \geq 1) = .8$ and

$$P(Y_2 = 1,\, Y_1 = 1) \;=\; P(Y_2 = 1,\, Y_1 = 1,\, X_1 = 1) \,+\, P(Y_2 = 1,\, Y_1 = 1,\, X_1 = 2)$$
$$=\; (.7)(.8) + (.1)(1) = .66$$

Next, it is easy to see that

$$P(Y_2 = 1,\, Y_1 = 0) \;=\; P(X_1 = 0,\, X_2 \geq 1) \;=\; (.2)(.2) = .04$$

so that $P(Y_2 = 1) = P(Y_2 = 1,\, Y_1 = 1) + P(Y_2 = 1,\, Y_1 = 0) = .7.$ Now

$$P(Y_3 = 1,\, Y_2 = 1,\, Y_1 = 0) \;=\; P(X_1 = 0, X_2 = 1, X_3 \geq 1) \,+$$
$$+\, P(X_1 = 0, X_2 = 2, X_3 \geq 1) \;=\; (.2)(.1)(.8) + (.2)(.1)(1) \;=\; .036$$

and

$$P(Y_3 = 1,\, Y_2 = 1,\, Y_1 = 1) \;=\; P(X_1 = 1, X_2 = 1, Y_3 = 1)$$
$$+\, P(X_1 = 1, X_2 = 2, Y_3 = 1) + P(X_1 = 2, X_2 = 1, Y_3 = 1) + P(X_1 = 2, X_2 = 2, Y_3 = 1)$$
$$= (.7)(.7)(.8) + (.7)(.1)(1) + (.1)(.1)(.8) + (.1)(.9)(1) = .56$$

Finally, pulling these results together we have

$$P(Y_3 = 1,\, Y_2 = 1) \;=\; .56 + .036 = .596$$

so that

$$P(Y_3 = 1 | Y_2 = 1) \;=\; \frac{.596}{.7} \;=\; .8514 \;\neq\; P(Y_3 = 1 | Y_2 = 1,\, Y_1 = 1) \;=\; \frac{.56}{.66} \;=\; .8485$$

which shows that $\{Y_k : \; k \geq 0\}$ is not Markovian.