Eric Slud 5/17/06, 4-6pm

Stat 650 In-Class Final Examination

Instructions. Do any 5 of the 6 problems. All count equally, 20 points each. 100 points is a perfect score. Justify your reasoning with results and theorems or previously solved problems from the class, wherever possible.

(1). Suppose that $X(t)$ is the number of customers in a **M/M/2** queueing system, with Poisson-process rate-λ arrivals and 2 servers each of whom serve customers in *Expon*(μ) time. Find $E_1(V_0)$ and $E_2(V_0)$, where V_0 is the random waiting time

$$V_0 \;=\; \inf\{\, t > 0 \;:\quad X(t-) \neq X(t) = 0\}$$

(2). A Markov chain $W(t)$ with nonnegative-integer-valued states has all of its nonzero off-diagonal $(k \neq j)$ transition intensities $q_{k,j}$ given by

$$q_{0,1} = q_{1,0} = q_{1,2} = 1$$

$$\text{for} \quad k \geq 2 \;: \qquad q_{k,k+1} \;=\; 3 \;\;, \qquad q_{k,k-1} \;=\; q_{k,k-2} \;=\; 1$$

(a) Show that if the embedded discrete-time chain Y_n for $W(t)$ is defined by $Y_n = W(T_n)$, then $Y_n - \sum_{i=1}^n I_{[Y_{i-1}=0]}$ is a martingale, where $T_0 = 0$ and for $n \geq 1$,

$$T_n \;=\; \inf\{\, t > T_{n-1} \;:\quad W(t) \neq W(T_{n-1}\}$$

(b) Find $E_1($ number of times $W(t)$ hits 0 before hitting $10\,)$.

(3). A mechanical system runs for a random time T distributed *Uniform*$(0,10)$ and is then instantaneously reapired with a new random lifetime distributed the same way and independent of the past. Each operating lifetime is called one 'cycle'. The cost of operating the system is \$10 per unit time for the first 7 time-units of life within each cycle, and \$15 per unit time for any time within cycle in excess of 7.

(a) Derive a renewal equation satisfied by the expectation of the cumulative cost $C(t)$ up to time t. (Let $C(t)$ denote the random cumulative cost and $c(t)$ its expectation.)

(b) Find the long-term cost per cycle and long-term cost per unit time incurred in running the system.

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(4). Consider the discrete-time Markov chain X_k with state-space $S = \{1, 2, 3, 4, 5\}$ and transition-matrix

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.5 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find $\lim_{n\to\infty} P^n_{32}$.

(b) Starting from state 4, find the expected total number of times that state 3 is hit.

(5). A small population changes according to the following dynamics. When the population size is 0, a new immigrant arrives after an $Expon(\alpha)$ waiting time. When the population size is 1, 2, or 3, each member of the population independently of the others produces an offspring after an $Expon(\lambda)$ waiting-time. However, at the instant when the population size would have grown to 4, all population members die (which makes the poulation size then equal to 0, never 4).

(a). Explain why the time-varying population size $X(t) \in \{0, 1, 2, 3\}$ is a continuous-time Markov chain, and find its intensity matrix.

(b). What proportion of the time, long-term, is the population size equal to 2 ?

(c). In what proportion of $0 \mapsto 0$ cycles does the population have two members for at least twice as long as it has 3 members ?

(6). Let $N_1(t)$, $N_2(t)$ be independent unit-rate Poisson counting processes for $t \geq 0$.

(a) What is the probability distribution of the number of jumps of $N_1(\cdot)$ between the first and second jumps of $N_2(\cdot)$?

(b) Let $W_1 < W_2 < \cdots$ be the successive jump-times for the process N_1 , and $V_1 < V_2 < \cdots$ the sequence of successive jump-times for N_2 . What is the conditional expectation of the number of intervals (W_k, W_{k+1}) falling completely within $[0, 10]$ which contain exactly one of the jump-times V_i, given only that $N_1(10) + N_2(10) = 22$?