Eric Slud 5/16/07 4–6:10pm

Stat 650 Final Examination

Instructions. On this Exam, each problem counts 20 points, and 100 are possible. The test is closed-book, but you may use 2 (two-sided) pages of notes for reference. Give details and justifications for your answers.

(1). Consider the Markov chain $\{X_n\}_{n\ge 0}$ on state-space $S = \{0, 1, 2, \ldots\}$ with $X_0 \equiv 0$ and transition matrix

$$P \;=\; \begin{pmatrix} .2 & .8 & 0 \\ .2 & 0 & .8 \\ .2 & .4 & .4 \end{pmatrix}$$

(a) Show that $Y_n = I_{[X_n=0]}$ for $n \ge 0$ is a HMC.

(b) Find the probability distribution of T_0 explicitly.

(c) Find $E_0(\sum_{k=1}^{T_0} I_{[X_k=1]})$ and give a discrete renewal equation satisfied by $E(\sum_{k=1}^{n} I_{[X_k=1]})$.

(2). In an M/M/1 queue with $q_{n,n+1} = \lambda$, $q_{n,n-1} = \mu\, I_{[n>0]}$ for all $n \in S = \{0, 1, 2, \ldots\}$, assume $\lambda < \mu$ and that the queue starts with initial (time-0) state chosen from the stationary probability distribution. Find the probability that at the time of the first new arrival, the server is idle, i.e., $X(S_1-) = 0$, where S_1 is the time of the first new arrival (i.e., first jump to a larger queue length) after time 0.

(3). A discrete-time Markov-chain with state-space $S = \{A, B, 1, 2, \ldots, 5\}$ is governed by the transition matrix

$$P \;=\; \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Find $\lim_{n\to\infty} P^{2n+1}_{Aj}$ for all $j \in S$.

(4). A discrete-time HMC X_k with nonnegative-integer states has transition matrix defined as follows: $P_{01} = 1$ and

$$\text{if} \quad k \ge 3: \ P_{k,k+2} = 1/2 \ , \quad P_{k,k-1} = 1/4 \ , \quad P_{k,k-3} = 1/4$$

$$\text{if} \quad k = 1, 2: \ P_{k,k+1} = P_{k,k-1} = 1/2$$

Recall that $T_0 = \inf\{k \ge 1 : \ X_k = 0\}$.

(a) Show that if $X_0 = i > 0$, then $X_{\min(k,T_0)}$ is a martingale.

Note: $X_{\min(k,T_0)}$ *is the process which behaves like* X_k *until it hits* 0*, but then stays stuck at* 0.

(b) For $0 < i < N$, calculate $P_i(X_k$ hits 0 before $N)$.

(c) Prove that X_k is recurrent.

(5). Let $N_1(t)$ and $N_2(t)$ be independent Poisson counting processes with respective rates $\lambda, \mu > 0$, and define a continuous-time HMC $X(t) = 2N_1(t) - N_2(t)$ with state-space S consisting of all integers.

(a) Give the intensity matrix for this HMC, and show that the embedded chain is a discrete-time (possibly biased) random walk on the integers.

(b) Give a necessary and sufficient condition in terms of λ and μ for this HMC to be recurrent.

(c) Give a differential equation for the transition probability $P_{0k}(t)$ as a function of time t. Do not try to solve it explicitly, but justify in terms of results given in the course that it does have a unique solution for each λ and μ. If λ and μ are such that the HMC is recurrent, what should be the limit of $P_{0k}(t)$ as $t \to \infty$?