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Stat 650 Sample Final Problems

Instructions. These problems are intended to be roughly of the difficulty and topic coverage as the problems you will be given on the Final Exam on Monday, May 16, 2006. although some of the ones given here are slightly more difficult that what I would give (e.g., # 2, 12). The main topic and idea for each problem is indicated in italics. The Final Exam will consist of about 5 or 6 problems like the single-topic problems, and I may include a short-answer (true/false style) section instwad of one of the problems.

The first five problems relate to the `M/M/1` queue with arrival rate $\lambda = 1$ and service rate $\mu = 3$, which is the name in applications (operations research and computer systems) of the Birth-and-Death process on the nonnnegative integers with transition-rates

$$Q_{j,j+1} = \lambda, \;\; Q_{j,j-1} = \mu \, I_{[j>0]}, \;\; Q_{j,j} = -\lambda - \mu \, I_{[j>0]}$$

(1). (*First-step analysis.*) Find $E_0(T_2)$ (the expected time for an empty queue until the first time there are 2 customers in the system).

(2). (*Martingale calculation of hitting probabilities*) Find $P_2(\text{hit 5 before 0})$ and $E_2(\text{time to hit } \{0, 5\})$. (*Hint: before hitting 0, the Markov Chain X_t is just a random walk in continuous time. You will need two different martingales in terms of X_t to solve this problem.*)

(3). (*Cycles in continuous-time Markov Chains*) Find the long-term proportion of time spent in state 3 with previous state having been 2. (*Hint: one way to do this is to use the fact that the Birth-Death process run in reverse time is the same birth-death process. But that is not the way I intended the problem to be done.*)

(4). (*Cycles in discrete-time Markov Chains*) Find the long-term proportion of transition steps spent in an even-numbered state.

(5). (*Conditioning*) Suppose that the owner of the service facility experiences a cost equal to the person-hours which customers have waited before their service begins. If the initial state of the queue is $X(0) = 2$ (one customer on line and one being served), then find the mean of the owner's accumulated cost up to the time of the first service completion.

(6). (*Stationary distributions of chains with transient states*) The continuous-time Markov chain X_t with time-t transition matrix $P(t)$ has state-space $S = \{A, B, 1, 2, 3\}$ and intensity-matrix

$$Q \;=\; \begin{pmatrix} -1.5 & 1 & .5 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 2 & -5 & 3 \\ 0 & 0 & 1 & 3 & -4 \end{pmatrix}$$

(a). Find the transition matrix P^* of the embedded discrete-time Markov-chain associated with Q.

(b). Find $\lim_{n\to\infty} (P^*)^n$ and $\lim_{t\to\infty} P_{B1}(t)$.

(7). (*Exponential r.v.'s and Poisson process*) If $N_1(t)$ and $N_2(t)$ are independent Poisson processes with respective rates 3 and 1, then find $P(N_1(t) \text{ hits } 6 \text{ before } N_2(t) \text{ hits } 3)$ numerically.

(8). (*Exponential r.v.'s*) Let $T_1, \ldots, T_5$ be independent random variables, $T_k \sim \text{Expon}(k)$. Find $P(\max(T_4, T_5) < \min(T_1, T_2, T_3))$.

(9). (*Martingales and hitting probabilities – discrete & continuous-time processes; compound Poisson or branching processes*) Suppose that ξ_j for $j \geq 1$ are *iid* random variables such that ξ_j falls with equal probabilities $1/3$ on each of the values $-1, 0, 1$, and suppose $N(t)$ is an independent Poisson process with rate 2. Define $Y_n = \sum_{j=1}^n \xi_j$, $X(t) = Y_{N(t)}$.

(a) Show that Y_n is a martingale.

(b) Show that $X(t)$ is a martingale in the sense that for each $s < t$,

$$E\Big(X(t) - X(s) \,|\, (N(u),\, u \leq s),\; (\xi_j,\; j \leq N(s)) \Big) \;=\; 0$$

and also that $X^2(t) - 4t/3$ is a martingale in the same sense.

(c) Find $P(X(t) \text{ hits -10 before } 15)$ and

$$E\Big(\inf\{t > 0 : \;\; X(t) \in \{-10,\, 15\}\,\}\Big)$$

(10). (*Poisson process*) Suppose that X_k for $k = 1, 2, \ldots$ is an *i.i.d.* sequence of random variables equal to 1 with probability 2/3 and equal to 2 with probability $1/3$, and let $N(t)$ be a Poisson rate λ process independent of $\{X_k\}$. Show that for each t, the counting processes

$$R_j(t) \;=\; \sum_{k=1}^{N(t)} I_{[X_k = j]} \quad , \qquad j = 1, 2$$

are independent Poisson processes, and find their rates.

(11). (*Formulation of intensity matrix and MC stationary distributions*) Suppose that a continuous-time system is defined with states $\{0, 1, 2, 3\}$ in terms of two independent Poisson processes $N_j(t)$ with respective rates $1, 2$ by $X(t) = 3N_1(t) - N_2(t) \bmod 4$. Show that the system is Markovian, and find its intensity matrix and its stationary probability of being in state 3.

(12). (*Recurrence criterion*) Suppose that $(X_k,\; k \geq 0)$ is an irreducible discrete-time homogeneous Markov chain with the nonnegative integers as states such that, for all $i > 10$, $\sum_{j>10} P_{ij} \leq 1/2$ and that $P_{i,j} > 0$ for all $i, j \leq 10$.

(a) Show that for any state $i > 10$,

$$E\Big(\min\{n \geq 0 : X_n \leq 10\}\Big) < \infty$$

(b) Conclude from (a) that $\{X_k\}_k$ is recurrent.

(13). (*Renewal equation*). Suppose that $X(t)$ is a Markov chain with states $S = \{0, 1\}$ with transition intensity matrix

$$Q = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$$

(a). Let $R(t) = \int_0^t I_{[X(s)=0]}\, ds$ denote the total amount of time spent in state 0 up to t. Show that $r(t) = E_0 R(t)$ satisfies a continuous-time renewal equation. (*Hint: let* $\pi(t) = E_1 R(t)$*, and use first-step analysis twice to get equations for each of* $r(t), \pi(t)$ *in terms of the other.*)

(b). Using either limit theorems for renewal functions or some other method, find $\lim_{t\to\infty} r(t)/t$.

(14). (*Poisson process on the plane*). Let $N(A)$ for $A \subset [0, \infty)^2$ be a homogeneous point process. Say what process characteristics (Markovian ? homogeneous-transition ? independent increment ? homogeneous or nonhomogeneous Poisson ?) the following two processes possess:

(a) $N_1(t) = N(\{(t_1, t_2) : \ 0 \leq \max(t_1, t_2) \leq t\}$

(b) $N_2(t) = N(\{(t_1, t_2) : \ 0 <= t_1 \leq t/2,\, 0 \leq t_1 \leq t\})$

Also, (c) What is the joint distribution of $N_1(1)$, $N_2(1)$? Find $P(N_1(1) > 1 \,|\, N_2(1) > 0)$.

(15). (*Classification of states, periodicity, absorption probabilities.*)

(16). (*Null-recurrence versus positive recurrence*) Consider the discrete-state Markov chain on $S = \{0, 1, 2, \ldots\}$ with transition probabilities $q_{0,1} = 1$, $q_{k,k+1} = 1 - q_{k,0} = (k/(k+1))^\alpha$. For which values of $\alpha > 0$ is the chain recurrent ? null-recurrent ? positive-recurrent ?

(17). (*Kolmogorov forward and backward equations for continuous-time Markov chains – solution techniques for small finite-state chains*)