Stat 650, E. Slud April 6, 2007

Handout on Invariant Vectors for Non-Recurrent HMC's

There was a question in class on Wednesday, April 4, about whether there could be nontrivial invariant vectors for countable-state irreducible non-recurrent HMC's. The purpose of this handout is put the question and its (affirmative) answer in context for Birth-Death chains on $S = \{0, 1, 2, \ldots\}$.

Recall that for general countable-state chains which are both irreducible and recurrent, we found in Chapter 3 that the vector with entries

$$x_i \;=\; E_0\Big\{ \sum_{k=1}^{T_0} I_{[X_k=i]} \Big\}$$

automatically provides an invariant S-indexed vector (i.e., a vector $\mathbf{x}$ such that $\mathbf{x}^{tr} P = \mathbf{x}^{tr}$) which has all strictly positive and finite entries, and which is summable if and only if the HMC is positive-recurrent.

Now restrict attention to the case of irreducible birth-death chains on $S = \{0, 1, 2, \ldots\}$ with transitions P satisfying

$$P_{ij} \;=\; q_i \, I_{[j=i-1]} \,+\, r_i \, I_{[j=i]} \,+\, p_i \, I_{[j=i+1]}$$

(Note that $q_0 = 0$ and irreducibility implies that p_0 and all $p_j, q_j > 0$ for $j \geq 1$.) We proved in class by direct calculation that the vectors $\mathbf{v}$ satisfying for $i \geq 1$

$$v_i \;=\; v_0 \cdot \prod_{j=0}^{i-1} \frac{p_j}{q_{j+1}}$$

are all stationary in the sense that $\mathbf{v}^{tr} P = \mathbf{v}^{tr}$. We now obtain a complete characterization of which of these birth-death HMC's are recurrent.

Consider the probabilities

$$\alpha_i \;=\; P_i\Big(X_k = 0 \quad \text{for some } \; k \geq 0 \Big) \, , \qquad i \geq 0$$

where $\alpha_0 = 1$ by definition. Then the birth-death property automatically implies α_i are nonincreasing in i, and since they are nonnegative, $\lim_{m\to\infty} \alpha_m = \alpha$ exists. We can make a direct argument to show that recurrence is equivalent to $\alpha > 0$. First, we know already by irreducibility of the chain that recurrence would imply $\alpha_m = 1$ for all $m \geq 1$, so that $\alpha = 1$. Conversely, if $\alpha > 0$, then we can by induction find a sequence of nonnegative integers $\{m_j\}_{j=0}^{\infty}$ with $m_0 = 1$ such that for all $j \geq 0$,

$$P_{m_j}(X_k \text{ hits } 0 \text{ before } m_{j+1}) \;\geq \alpha/2 \qquad (*)$$

(To do this, noting that all $\alpha_i \geq \alpha > 0$, we can for given m_j find m_{j+1} satisfying the desired property since the limit of the left-hand side of (*) as $m_{j+1} \to \infty$ is $\alpha_{m_j} \geq \alpha$. Finally, using (*) repeatedly, we conclude

$$P_{m_1}(\text{ hit 0 before } m_N) \geq 1 - (1 - \frac{\alpha}{2})^N$$

which implies by letting $N \to \infty$ that the probability starting from $m_0 = 1$ of the chain returning to 0 in finitely many steps is 1. The conclusion from $\alpha > 0$ is thus that $P_0(T_0 < \infty) = 1$, and the chain is recurrent.)

Now consider the sequence of α_m again: in the transient case, we have seen that $\alpha_1 < 1$ and $\alpha_m \searrow 0$, and first-step analysis shows immediately that for $i \geq 1$,

$$\alpha_i = q_i \alpha_{i-1} + r_i \alpha_i + p_i \alpha_{i+1}$$

This equation immediately implies that

$$w_i \equiv \alpha_i - \alpha_{i+1} \;=\; \frac{q_i}{p_i}\, w_{i-1} \;=\; \prod_{j=1}^{i} \frac{q_j}{p_j} \cdot w_0$$

Therefore

$$\alpha_1 - \alpha_{m+1} \;=\; (1 - \alpha_1) \sum_{i=1}^{m} \prod_{j=1}^{i} \frac{q_j}{p_j}$$

Letting $m \to \infty$ and substituting the limiting value 0 for α_m, we find

$$\alpha_1 \;=\; (1 - \alpha_1) \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{q_j}{p_j}$$

and in particular the last summation must be finite.

So we have found a condition, namely that

$$\sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{q_j}{p_j} < \infty$$

which implies that (actually, is equivalent to the assertion that) the birth-death chain is transient. There are many ways for this to happen: most simply, in the biased-up random walk case $q_j/p_j \equiv \rho < 1$ for all $j \geq 1$. But many more balanced cases such as $q_j/p_j = (j/(j+1))^2$ also result in transient chains. Yet all of them have invariant (non-summable) vectors, as we remarked at the beginning of these pages.