STAT 700 Final Exam, 8-10am, Dec. 17, 2022

Instructions. You may use a single notebook page of memory aids for the exam. You may use a calculator if you like, but you need not give a numerically simplified answer in any numerical problem. Explain your reasoning in words, referring to standard results whenever you can state them. **The exam has a possible 100 points: do problems 1–3 and either 4 or 5.**

(1). (*25 points*) Suppose that a sample $\underline{X} = (X_1, \ldots, X_n)$ has density $f(x, \theta) = I_{[x \geq 0]} \cdot \theta/(1+x)^{\theta+1}$, where $\theta > 2$ is an unknown parameter.

(a). Show that this is an exponential family of densities, and find the sufficient statistic and its mean and variance.

(b). Find $E_\theta(X+1) = \psi(\theta)$. Is there a UMVUE for this parameter $\psi(\theta)$? (You do not have to calculate it, just show whether it exists.)

(c). Is there an estimator with expectation $\psi(\theta)$ that exactly attains the Cramér-Rao (Information Inequality) lower bound for $\psi(\theta)$? Show why or why not, either by calculation or theoretical argument.

(2). (*25 points*) Suppose that a sample of 20 *iid* observations X_i each have the density $f(x, \theta) = e^{-(x-\theta)} I_{[x \geq \theta]}$, where $\theta \geq 0$. Show that there is a UMP test of $H_0 : \ \theta \leq 2$ versus $H_A : \ \theta \in (2, 4]$, and give an expression (as explicit as you can, but not numerically reduced) for its rejection region.

(3). (*25 points*) Suppose that X_i for $i = 1, \ldots, n$ is a sample of Gamma$(3, \lambda)$ random variables on the parameter space $\lambda \in (0, \infty)$, and that we want to find a Bayes optimal estimator of λ based on these data for the loss function $L(\lambda, a) = \lambda \cdot (\lambda - a)^2$ using the prior density $\pi(\lambda) \sim$ Gamma$(2, 2)$.

(a). (*10 points*) Find the posterior density of λ, and explain why it can depend on $\underline{X}$ only through the sufficient statistic $\bar{X}$ for λ.

(b). (*5 points*) Find the Bayes posterior risk function $E(\int_0^\infty L(\lambda, g(\bar{X}))\, \pi(\lambda)\, d\lambda \,|\, \underline{X})$ for the estimator $g(\bar{X})$.

(c). (*10 points*) Using the same prior density as in (a), find the Bayes-optimal estimator of λ, i.e. the estimator $g(\bar{X})$ minimizing

$$r_\pi(g(\bar{X})) \; = \; E(\int_0^\infty L(\lambda, g(\bar{X}))\, \pi(\lambda)\, d\lambda)$$

Do problem 4 or 5, not both !

(4). (*25 points*) Suppose that a decision problem has parameter θ, data $\underline{X}$, loss function $L(\theta, a)$ and risk function $R(\theta, g)$ for (possibly randomized) decision rule $g(\cdot)$. Show that if the decision rule $\delta(\cdot)$ is admissible and also has the property that $R(\theta, \delta) = c$ is a constant, not varying with θ, then $\delta(\cdot)$ is minimax.

(5). (*25 points*) A single binary random variable X is observed under one of two probability models parameterized by $\theta \in \{1, 2\}$, where

$$p_{\theta=1}(X = 0) \;=\; 1 - p_{\theta=1}(X = 1) \;=\; \frac{1}{4} \;, \quad p_{\theta=2}(X = 0) \;=\; 1 - p_{\theta=2}(X = 1) \;=\; \frac{1}{2}$$

The decision problem has two possible pure actions $a = 1, 2$, and a class of randomized decision rules ϕ_b for $b \in [0, 1]$ is defined by

$$\phi_b(x) \;=\; P(a = 1 \,|\, X = x) \;=\; b\, I_{[x=0]} \;+\; (1 - b)\, I_{[x=1]}$$

The loss function in this problem is given by

$$L(\theta, a) \;=\; I_{[\theta=1,\, a=2]} \;+\; 2\, I_{[\theta=2,\, a=1]}$$

(a). Find the risk function $R(\theta,\, \phi_b)$ as a function of b.

(b). Use the result of (a) to show that there is a unique best test ϕ_b in this problem, and find it.

Do problem 4 or 5, not both !