STAT 700 August 26, 2024

# First Problem Set in STAT 700, F24

These 6 problems mostly relate to prerequisite material for Stat 700. The due-date is Wednesday, September 4, 2024, 11:59pm, for a pdf-format upload to the course ELMS page.

**(Q1)** Suppose that a sequence of independent random variables $X_i,\ i = 1, 2, \ldots$, has distributions

$$X_{2k} \sim \text{Unif}[0, 2] \quad \text{and} \quad X_{2k-1} \sim \text{Expon}(1) \quad \text{for} \quad k = 1, 2, \ldots$$

Then give the density for the limiting distribution of $\frac{1}{\sqrt{2n}} \sum_{i=1}^{2n} (X_i - 1)$.

**(Q2)** Suppose that a sequence of independent random variables $\{Y_i\}_{i=1}^n$ have a *location-scale* family of densities, i.e., that for some $\mu \in \mathbb{R}$, $\sigma > 0$ and a known density $f_0$ on the real line such that $Y_i \sim \frac{1}{\sigma} f_0((x - \mu)/\sigma)$. Prove that for any bounded continuous function $g : \mathbb{R}^n \mapsto \mathbb{R}$,

$$E(g(Y_1, \ldots, Y_n)) = \int_{\mathbb{R}^n} g(\sigma z_1 + \mu, \ldots, \sigma z_n + \mu)\, f_0(z_1) f_0(z_2) \cdots f_0(z_n)\, dz_1 \cdots dz_n \qquad (1)$$

Use (1) to conclude that if $(Z_1, \ldots Z_n)$ is an n-tuple of independent random variables with density $f_0$, then

$$(\sigma Z_1 + \mu, \ldots, \sigma Z_n + \mu, \ldots) \stackrel{\mathcal{D}}{=} (Y_1, \ldots, Y_n) \qquad (2)$$

**(Q3)** Suppose that $V_1, V_2$ are independent random variables, each with density $f_V(v) = 2v\, I_{[0<v<1]}$, and let $W_1 = \min(V_1, V_2)$, $W_2 = \max(V_1, V_2)$.

(a). Reason from first principles to find $P(W_1 \le s,\, W_2 > t)$ for all $0 \le s \le t \le 1$.

(b). Use part (a) to find the joint density of $W_1, W_2$.

(c). Use the density in (b) to find $E(W_1^2 \,|\, W_2)$.

(d). Extend the reasoning in parts (a), (b) to find explicitly the joint density of $W_1 = \min(V_1, \ldots, V_m)$, $W_2 = \max(V_1, \ldots, V_m)$, if $V_1, \ldots V_m$ are independent random variables with density $f_V$, where $m \ge 3$.

**(Q4)** Use formula (1) to prove that the unknown parameters $(\mu, \sigma)$ are *identifiable* in the sense defined in class, from a sample of observations $Y_1, \ldots Y_n$ from the location-scale family in **(Q2)**.

**(Q5)** *Jacobian change-of-variable.* Suppose that $X_1, \ldots, X_4$ are independent Expon(1) distributed random variables. Find the joint density of $(Y_1, \ldots, Y_4)$ defined by

$$Y_1 = 3X_1, \quad Y_2 = 2X_2^2, \quad Y_3 = X_2 + X_3, \quad Y_4 = X_1^4 + 2X_4$$

**(Q6)** *Linear Algebra.* Let the $5 \times 5$ matrix $A$ be defined by $A = c\,\mathbf{1}\,\mathbf{1}^{tr} + I_5$, where $c > 0$ is a constant, $\mathbf{1}$ is the 5-dimensional vector with all entries 1, and $I_5$ is the $5 \times 5$ identity matrix. Find a formula for $A^{-1}$ giving its entries explicitly in terms of $c$. *Hint: spectral decomposition.*