STAT 700 In-Class Test 2022, Nov. 2, 2022, 9–9:55am

Instructions. Do all 3 problems. Point values of problem parts are indicated and sum to 120; the maximum possible score is 100. Explain your reasoning, and justify your steps by theorems and facts proved in class.

(1). Consider the following decision-theory setting, with no data. The parameter and action spaces are both $\Theta = \mathcal{A} = \{0, 1\}$, and the loss function is

$$L(\theta, a) \;=\; 2 \cdot I[\theta = 0, a = 1] \;+\; I_[\theta = 1, a = 0]$$

For $0 \le b \le 1$, let γ_b denote the randomized decision rule that chooses action $a = 0$ with probability b and action $a = 1$ with probability $1 - b$. Let π denote the prior distribution on Θ that assigns probability $\pi(0)$ to $\theta = 0$ and probability $\pi(1) = 1 - \pi(0)$ to $\theta = 1$.

(a). (*14 points*) Find the unique minimax randomized decision rule.

(b). (*14 points*) For fixed $\pi(0) \in (0, 1)$, find the unique randomized Bayes-optimal decision rule with respect to the loss function L and prior π.

(c). (*10 points*) Are there any randomized decision rules γ_b that are not admissible ?

(2). Let $\underline{X} = (X_1, \ldots, X_n)$ be an *iid* sample from $f(x, \alpha, \lambda) \equiv (\lambda^\alpha / \Gamma(\alpha))\, x^{\alpha-1}\, e^{-\lambda x}\, I_{[x>0]}$, where the parameters $\alpha, \lambda > 0$.

(a). (*12 pts*) Give the natural (minimal) sufficient statistic and natural parameter space.

(b). (*25 pts*) Under the parametric family $f(x, \alpha, \alpha)$ restricted by $\lambda = \alpha$, show that the joint density is of exponential-family form and find the natural (minimal) sufficient statistic and natural parameter space $\mathcal{E}$. Is this a curved exponential family ? What is its rank ?

(3). Let $X_1, \ldots, X_n$ be an *iid* sample from $f(x, \theta) = (2x/\theta^2) \cdot I_{[0 \le x \le \theta]}$, with $\theta > 0$.

(a). (*16 pts*) Show that there is a scalar-valued sufficient statistic in this problem, and find its distribution function.

(b). (*15 pts*) Show the sufficient statistic you found in (a) is complete.

(c). (*14 pts*) Find the UMVUE for θ^2. *Hint: The random variables X_i form a scale family, i.e. X_i/θ has a distribution that does not depend on θ.*