STAT 701 HW3 Solutions, 3/13/23

(1). (#5.4.2) Assume (A0)–(A3) and (A4′). The hint in the problem is proved by upper-boundng via the triangle inequality

$$\sup_{|t-\theta_0|\le\epsilon_n} |\frac{1}{n}\sum_{i=1}^n (\frac{\partial\psi}{\partial\theta}(X_i,t) - \frac{\partial\psi}{\partial\theta}(X_i,\theta_0))| \;\le\; \frac{1}{n}\sum_{i=1}^n \sup_{|t-\theta_0|\le\epsilon_n} |\frac{\partial\psi}{\partial\theta}(X_i,t) - \frac{\partial\psi}{\partial\theta}(X_i,\theta_0))|$$

and taking expectations, using the fact that the summands on the right-hand side are *iid*. By the Dominated Convergence Theorem, (A.4) then holds if and only if the expectatino of the left-hand side of the displayed equation convergence to 0, and this is true by continuity of $\partial\psi(x,\cdot)/\partial\theta$ since the summands on the right-hand side converge with probability 1 to 0 as $\epsilon_n \to 0$: the RHS expectations go to 0 because the terms are dominated by $M(X_i,\theta_0)$ which have finite expectation.

(2). (#5.4.3) There would have been no harm in this exercise if $d\mu(x)$ were replaced by dx on the Euclidean observation-space where X_i take their values. As mentioned below (5.4.20) in the book, the statement in (5.4.30) is simply that

$$\int \frac{\partial}{\partial\theta}\,(\psi(x,\theta)\,p(x,\theta))\,dx \;=\; 0 \qquad \text{for all} \quad \theta$$

In the problem's Hint, the right-hand side of the displayed equation is identically 0 for all finite a, b by (5.4.31), which implies that

$$\int\int_a^b \frac{\partial}{\partial\theta}\,(\psi(x,\theta)\,p(x,\theta))\,d\theta\,dx \;\equiv\; 0$$

The condition in (A.6′) implies (because any bounded $[a, b]$ can be covered by finitely many intervals $(\theta - \delta(\theta),\, \theta + \delta(\theta)))$ that one can switch the order of integration in the last expression, by Fubini's Theorem. Therefore, for all finite a, b

$$\int_a^b\int \frac{\partial}{\partial\theta}\,(\psi(x,\theta)\,p(x,\theta))\,dx\,d\theta \;\equiv\; 0$$

which implies buy the funcamental theorem of calculus (differentiate d/db) that the expression $\int\{\cdot\}\,dx$ in the last displayed equation is identically 0, and the book already mentioned that this is equivalent to trhe first line of (5.4.30).

(3). (#5.4.10) (a). $A_n = -n^{-1}\sum_{i=1}^n \frac{\partial^2}{\partial\theta^2}\log f(X_i,\theta_0)$ is consistent for $I(\theta_0)$ according to the Law of Large Numbers and (5.4.30). If you use (A.4), then $|A_n - \hat{I}| \cdot I_{[|\hat\theta-\theta_0}| \le$

$\epsilon_n] \to 0$ in probability, and $P(|\hat\theta - \theta_0| > \epsilon) \to 0$ by (A.5) if $\epsilon \to 0$ slowly enough; if you use (A.4'), then $E|A_n - \hat I| \to 0$ by the Dominated Convergence Theorem.

(b). Since $\sqrt{n}\,(\hat\theta - \theta_0) \to \mathcal{N}(0,\, 1/I(\theta_0))$, it follows that $P(\sqrt{n}\,(\hat\theta - \theta_0) \le \Phi^{-1}(1-\alpha)/\sqrt{I(\theta_0)}) \to 1-\alpha$ and for all $\epsilon > 0$, $P(|1/\sqrt{I(\theta_0)} - 1/\sqrt{I(\hat\theta)}| \ge \epsilon) \to 0$. Then by Slutsky's Theorem and continuity of $\Phi(\cdot)$,

$$P(\sqrt{n}\,(\hat\theta - \theta_0) \le \Phi^{-1}(1-\alpha)/\sqrt{I(\hat\theta)}) \to 1-\alpha \quad \Longrightarrow \quad P(\theta_0 \ge \hat\theta - \Phi^{-1}(1-\alpha)/\sqrt{I(\hat\theta)}) \to 1-\alpha$$

(4). (#5.4.14) (a). In this Inverse Gaussian problem, we must use the identity

$$\Big(\frac{1}{2\pi\, x^3}\Big)^{1/2} \int_0^\infty \exp\big(-\,\lambda[\frac{x}{2\mu^2} + \frac{1}{2x}]\big)\, dx \;\equiv\; \frac{1}{\lambda^{1/2}}\, e^{-\lambda/\mu} \qquad (*)$$

Inverse-Gaussian is a canonical exponential family in the parameter λ, with $T(x) = -x/(2\mu^2) - 1/(2x)$. So the family is MLR, decreasing in $\sum_{i=1}^n T(X_i)$, and the one-sided test for $\lambda = \lambda_0$ versus $\lambda < \lambda_0$ rejects for large values of $\sum_{i=1}^n [X_i/(2\mu^2) + 1/(2X_i)]$.

(c). In (*), differentiate with respect to λ once to learn that $E(T(X)) = -1/\mu - 1/(2\lambda)$ and $E(T^2(X)) = 3/(4\lambda^2) + 1/\mu^2 + 1/\mu\lambda$, so that $\mathrm{Var}(T(X)) = 1/(2\lambda^2)$. Then apply the Central Limit Theorem to find the asymptotic approximate upper $1-\alpha$ quantile of $\sum_{i=1}^n [X_i/(2\mu^2) + 1/(2X_i)]$ as

$$n(1/\mu + 1/(2\lambda_0)) + \Phi^{-1}(1-\alpha)\,\sqrt{n/(2\lambda_0^2)} \qquad (**)$$

(e). The Rao Score rejection and large-sample UMP Neyman-Pearson regions are the same in the case where θ is univariate and the density is a canonical exponential family. These tests reject when: $\sum_{i=1}^n [X_i/(2\mu^2) + 1/(2X_i)] \;>\; (**)$.

(d). The Wald test is based on the MLE, which in the inverse-Gaussian family is $\hat\lambda = (n/2)\,/\,\sum_{i=1}^n(-1/\mu - T(X_i))$. This test rejects for $\hat\lambda$ less than $\lambda_0 \;-\; \Phi^{-1}(1-\alpha)\,\sqrt{2\lambda_0^2\, n}$, since $I(\lambda_0) = \mathrm{Var}(T(X)) = 1/(2\lambda_0^2)$.

(A). As stated several times in class, the MLE for μ is $\bar X$ in the $\mathcal{N}(0,1)$ model and an MLE is the sample median $\tilde\mu^{med}$ in the double-exponential model. Evidently, $\sqrt{n}\,(\bar X - \mu)$ is asymptotically normal with mean 0 and variance $\mathrm{Var}(X_1)$, which is 1 in the normal model and 2 in the double-exponential. As covered in class in problem 5.4.1(e), $\sqrt{n}\,(\tilde\mu^{med} - \mu)$ is asymptotically normal with mean 0 and variance $= 1/(4(f_X(\mu))^2)$ which is equal to 1 in the doubple-exponential model and $1/(4/2\pi) = \pi/2$ in the normal model. Thus the ARE of $\tilde\mu$ versus $\bar X$ is $2/\pi$ in the normal model, and the ARE of $\bar X$ versus $\tilde\mu$ is 1/2 in the double-exponential model.

(B). We are testing for the rate-parameter λ in the Expon(λ) model, which is MLR for statistic $\bar{V}$, and the UMP test for $H_0 : \lambda \le \lambda_0$ versus $H_1 : \lambda > \lambda_0$ rejects when $\bar{V} \le c$. Similarly, the one-sided UMP test in the other direction rejects when $\bar{V} \ge c'$. In part (a), the use of exact quantiles for the Gamma($n, 1$) random variable $n\lambda_0 \bar{V}$ under the hypothesis $\lambda = \lambda_0$ gives the injterval which is the intersection of the two acceptance regions

$$\lambda_0 \in (\text{Gamma}(n,1)^{-1}(\alpha/2),\ \text{Gamma}(n,1)^{-1}(1-\alpha/2))/(n\bar{V}) \ = \ (0.798,\ 1.221)/\bar{V}$$

In (b), the posterior for λ is Gamma($n+2$, $n\bar{V} + 0.1$), so the credible interval is

$$\lambda_0 \in (\text{Gamma}(n+2,1)^{-1}(\alpha/2),\ \text{Gamma}(n+2,1)^{-1}(1-\alpha/2))/(n\bar{V}+0.1)$$

or $(0.827,\ 1.258)/(\bar{V} + 0.1/60)$. Finally, in part (c), with the MLE $\hat{\lambda} = 1/\bar{V}$, we have $\sqrt{n}\,(\hat{\lambda} - \lambda_0) \approx \mathcal{N}(0,\ \lambda_0^2)$, so the interval is based on $\sqrt{n}(\hat{\lambda}/\lambda_0 - 1) \in (-\Phi^{-1}(1-\alpha),\ \Phi^{-1}(1-\alpha))$, or $\lambda_0 \in (1 \pm z_{.05}/\sqrt{n})^{-1}/\bar{V} = (0.825, 1.270)/\bar{V}$.

Coverage for the interval in (a) is exactly 90% as is therefore most accurate; coverage *would* be exact if λ were randomly generated according to the indicated prior, but it should still be fairly close to exact even for fixed λ_0. Coverage in part (c) is only approximate, and is likely the least accurate.