Final Examination, Stat 701, 8-10am, 5/17/23

The exam is closed-book: you may use one or two single-sided formula sheets as a memory aid, but no other books or papers. You may use a calculator, but you need **not** give a numerically simplified answer in any numerical problem. Explain your reasoning in words, referring to standard theorems whenever you can state them. The exam has a possible 100 points.

(1). A large random sample $X_1, \ldots, X_n$ of size n is assumed to follow the density $f(x, \theta) = \theta\,(1+x)^{-\theta-1}$ for all $x > 0$, where $\theta > 0$ is an unknown parameter.

(a). (*20 points*) As a function of θ, find the Asymptotic Relative Efficiency of the Generalized Method of Moments estimator of θ based on the function $g(x) = I_{[x>e-1]}$, i.e., the estimator solving the estimating equation $\sum_{i=1}^{n} \{I_{[X_i>e-1]} - e^{-\theta}\} = 0$.

(b). (*15 points*) Find the estimating equation estimator for θ with smallest asymptotic variance. (Give both the estimator and its asymptotic variance.)

(2). A sample $\{(Y_i, Z_i)\}_{i=1}^{n}$ of *iid* pairs of nonnegative integers satisfy $Y_i \sim \text{Poisson}(r\,\lambda)$, $Z_i \sim \text{Poisson}(\lambda)$, with Y_i and Z_i independent, where r, $\lambda > 0$ are unknown parameters.

(a) (*27 points*) Find the Rao-Score and Wald Tests (rejection regions) of the hypothesis $H_0 : r = 1$ versus the general alternative $H_A : r \neq 1$, with approximate size $\alpha = 0.05$.

(b) (*15 points*) Find the limiting power of the Wald test in part (a), for large n, against the simple alternative $H_1 : r = 1 + 1/\sqrt{n}$, $\lambda = \lambda_0$. (It should not depend on n.) The answer will depend on the true $\lambda = \lambda_0$, but λ was treated as **unknown** when you formulated your test in (a).

(c) (*8 points*) Does your answer in (b) tell you anything about the power of the Rao-Score test versus H_1 ?

(3). A large sample of 1000 binary observations $Z_i \sim \text{Binom}(1, \theta)$ is collected, yielding $1000 \cdot \bar{Z} = \sum_{i=1}^{1000} Z_i = 360$. The parameter $\theta \in (0,1)$ is unknown, but you believe that the appropriate prior distribution for it is $\text{Beta}(\alpha, \beta)$ with $\alpha = 3$, $\beta = 5$.

(a). (*15 points*) Explain why the Bernstein-von Mises Theorem applies, and what it says about the approximate posterior distribution for θ given $(Z_1, \ldots, Z_{1000})$. Would your answer change if the prior were $\text{Beta}(\alpha, \beta)$ with $\alpha, \beta \in (1, 5)$ unknown ?

(b). (*12 points*) Find a two-sided equal-tailed 95% Bayesian credible interval for θ based on the exact posterior for the prior with $\alpha = 3, \beta = 5$; and find the approximate Bayesian credible interval using the posterior you found in (a).