Additional Stat 701 Sample Problems for Final, S14

(I). (*Wald, Score and GLRT Tests*) A sample of observations $X_1, \ldots, X_n$ from a Gamma(α, β) density

$$f(x, \alpha, \beta) \;=\; \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}\, I_{[x>0]}$$

For all problem parts below, you may use the formula for the Fisher Information matrix (which we found in Problem (I) in HW4)

$$I(\alpha, \beta) \;=\; \begin{pmatrix} 1/\beta^2 & (.4228 - \ln\beta)/(\alpha\beta) \\ (.4228 - \ln\beta)/(\alpha\beta) & (.8237 - .8456 \ln\beta + \ln^2\beta)/\alpha^2 \end{pmatrix}$$

You may express your answers in terms of I_{11}, I_{21}, I_{22} if you wish.

(a). Find the Rao Score test for $H_0 :\; \alpha = 1$ against the general (two-sided) alternative, including the approximate large-sample cutoff for your test statistic to give a size .05 test.

(b). Although there is no closed-form MLE in this problem, there is an easily derived equation from the β score equation which expresses $\hat\alpha$ uniquely in closed form in terms of $\hat\beta$. Find this expression.

(c). Use the expression in part (a) to give the Generalized Likelihood Ratio Test of size .05 (with large-sample approximate cutoff) in terms of $\hat\beta$ for $H_0 :\; \alpha = 1$ versus the general alternative.

(II). (*GMOM vs ML based estimators*) A sample $Z_1, \ldots, Z_n$ of $\mathcal{N}(\mu, \sigma^2)$ (with both parameters unknown) is observed, and it is desired to estimate a target parameter defined as the *signal-to-noise ratio* $\gamma = \mu/\sigma$ (which is the reciprocal of the more common *cofficient of variation*).

(a). Find the asymptotic variance of the substitution estimator for γ based on the Maximum Likelihood Estimators.

(b). Find the limiting distribution of the Generalized Method of Moments Estimator of γ based on $h(Z_i) = I_{[Z_i>0]}$.

(III). (*Univariate estimating equation*) Suppose you observe a data sample $W_1, \ldots, W_n$ from Expon(β_0). Define an estimator $\tilde{\beta}$ as the solution to the estimating equation

$$\sum_{i=1}^{n} W_i \, (\beta \, W_i \, - \, 2) \; = \; 0$$

(a). Explain how you know that the estimator is consistent, and

(b). use what you know about estimating equations to find the asymptotic distribution of $\sqrt{n}\,(\tilde{\beta} - \beta_0)$.

(IV). (*CI based on Monte Carlo, asymptotics or bootstrap*) Suppose that you have observed $X_1, \ldots X_n$ from what you think is a Logistic(μ, σ) location-family density $f(x, \mu, \sigma) = \sigma^{-1}\, e^{(x-\mu)/\sigma} / (1 + e^{(x-\mu)/\sigma})^2$, and that you estimate the parameters by the method of moments. Explain how you could calculate a confidence interval using each of:

(i) an asymptotically pivotal quantity together with reference distributions given by the central limit theorem;

(ii) a Monte Carlo ('parametric bootstrap') simulation;

(iii) a nonparametric bootstrap, using the "bootstrap percentile" idea.

Explain which of these confidence intervals make sense to use respectively in the four situations where $n = 300$ and when $n = 30$, when the data do and do not truly come from the Logistic density family. Which type of interval do you think is the best choice in the four settings (two difference sample-sizes crossed with two underyling data-generating mnechanisms) ?