Sample Problems for Stat 701 In-Class Final, Spring 2009

Instructions. *Each* of the following is a problem roughly of the sort that might be on the Final. There will likely be five such problems in all.

(I) Prove consistency of the MLE $\hat{\vartheta}$ in the case where the parameter space $\Theta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_K\}$ is finite, based on large-sample *iid* data $\{X_i\}_{i=1}^n$ following one of the distinct densities $f(x, \vartheta_j)$ where $j = 1, \ldots, K$. For consistency, do you need to assume anything else about the densities ? Can you say anything about the large-sample distribution about the finite-valued MLE $\hat{\vartheta}$?

(II) State a version of the local uniqueness, consistency and asymptotic normality assertions about MLE's under 'suitable regularity conditions' [you need not give the regularity conditions] on the data densities $f(x, \vartheta)$, $\vartheta \in \Theta$. Explain why these results are useful in practice only when a consistent estimator (maybe a very slowly convergent one !) for the true ϑ is available from another source. If you knew instead – as is true for estimation in natural canonical exponential – that the MLE is unique and globally consistent and that the log-likelihood was everywhere concave, how would that help ?

(III) Suppose you observe 'matched pairs' data $X_{ij} \sim \mathcal{N}(\mu+\beta_j+\alpha_i,\, \sigma^2)$ where $1 \le i \le n$ $j = 1, 2$. Find a pivotal quantity for creating a Confidence Interval for the single unknown parameter $\beta_2 - \beta_1$ and give a 95% two-sided confidence interval based on such data for this parameter. *Note that this is possible, and not very difficult, even though consistent estimation of the parameters* α_i *is impossible in this setting.*

(IV) (a). Compare the asymptotic variance for estimating μ based on *iid* $\mathcal{N}(\mu, \sigma^2)$ data $\{Y_j\}_{j=1}^n$ when σ^2 is known to when it is not known. (Make sure you understand why they are actually the **same** in this case !)

(b). Explain why the same phenomenon occurs for data $Y_j \sim \frac{1}{\sigma} f_0((y-\mu)/\sigma)$ where f_0 any twice continuously differentiable and symmetric (i.e., even) and everywhere positive density function with $\int (f_0'(y))^2 / f_0(y)\, dy < \infty$.

(V) Suppose that observations X_i, $1 \le i \le n$, are *iid* $\mathcal{N}(\mu, \sigma^2)$.

(a). Give the Neyman-Pearson test $\varphi_\sigma(\mathbf{x})$ at significance level α for testing $H_0 : \mu = 0$ versus $H_1 : \mu = 1$.

(b). If σ is unknown but has a known prior distribution, i.e., if $\sigma \sim \pi(\cdot)$ is an unobserved r.v. and X_i conditionally given σ are *iid*, then find the form of the optimal (Neyman-Pearson) test of H_0 versus H_1.

(c). Same question as (b), if now $\sigma_i \sim \pi(\cdot)$ are *iid* but unobserved and conditionally given $\{\sigma_i\}_{i=1}^n$, the independent r.v.'s $X_i \sim \mathcal{N}(\mu, \sigma_i^2)$ are observed.

I might also have asked in one or another parts for an expression for power.

(VI). Based on data $X_i,\ 1 \le i \le n$ reduced to $T_1 = \sum_{i=1}^n X_i$ and $T_2 = \sum_{i=1}^n \log(X_i)$, find an asymptotic goodness of fit test to the *Expon*(ϑ) family of distributions (indexed by unknown $\vartheta > 0$) along with rejection region, against Gamma(β, ϑ) alternatives with $\beta \ne 1$. You may take as known that *digamma*$(1) \equiv \Gamma'(1)/\Gamma(1) = \int_0^\infty \log(x)\, e^{-x}\, dx = -0.57722$, and that

$$\int_0^\infty \begin{pmatrix} \log x \\ x \end{pmatrix}^{\otimes 2} e^{-x}\, dx = \begin{pmatrix} 1.97811 & 0.42278 \\ 0.42278 & 2.0 \end{pmatrix}$$

Hint: *the point is that the Rao Score test is quite feasible to find in this example rather explicitly in closed form, but the MLE and LR test could be found only implicitly in terms of an iteratively computed log-likelihood maximizer.*

(VII). *Delta Method, or Generalized Method of Moments problem, e.g., asking for relative efficiency versus MLE in a specific setting.*

(VIII). *Calculation and interpretation of power versus alternatives* $\vartheta_0 + c/\sqrt{n}$ *in a specific setting.*

(IX). χ^2 *Goodness of Fit test in specific multinomial setting, perhaps with an estimated parameter.*