Sample Stat 701 In-Class Test, Spring 2009

Instructions. *Each* of the following problem parts is intended as a problem roughly of the sort that might be on the Test. There will likely be three such problems in all.

(I) [*Generalized Moment Estimators vs. MLE, from Summer 2006 Qualifying Exam*] Consider a sample $Z_1, \ldots, Z_n$ from an Exponential(λ) density with unknown $\lambda > 1$, and define an estimator $\tilde{\lambda}$ as the Method of Moments Estimator of λ based on the variables $W_i \equiv e^{Z_i/2}$. Find an explicit form for $\tilde{\lambda}$; find the large-sample limiting distribution of $\sqrt{n}(\tilde{\lambda} - \lambda)$; and find the relative efficiency of $\tilde{\lambda}$ as an estimator of λ.

(I′) Let $(x_1, \ldots, x_n)$ be a sample from a population with unknown parameter $\theta \in \mathbf{R}$ in probability density of the form

$$2|x-\theta|^3 \exp\{-(x-\theta)^4\} \quad , \qquad \text{all} \quad x \in \mathbf{R}$$

(i) Find the method of moments estimator of θ and after an appropriate normalization, its nondegenerate limiting distribution. (**Hint:** Recall the identities $\Gamma(y+1) = y \cdot \Gamma(y)$ for $y > 0$, and $\Gamma(1/2) = \sqrt{\pi}$).

(ii) Calculate the Fisher information on θ and the asymptotic efficiency of the method of moments estimator.

(II) [*Estimators in Exponential Families*] Give the necessary and sufficient condition for the Poisson regression exponential family MLE's to exist. Are they unique ? consistent ? efficient ? Poisson Regression is the model saying that Y_i are independent nonnegative integer-valued and $\mathbf{Z}_i$ a set of fixed known p-vectors such that $Y_i \sim \text{Poisson}(\exp(\beta' \mathbf{Z}_i))$. Assume that the set of vectors $\{\mathbf{Z}_i\}_{i=1}^n$ spans $\mathbf{R}^p$. *The idea is to formulate this model as a natural and canonical exponential family model and check the conditions of theorems in Bickel and Doksum about MLE's for such models.*

(III) [*Delta Methods and Approximations*]

(a). Suppose that $X_i \sim \text{Expon}(\lambda)$, $i = 1, \ldots, n$, and let $\hat{\lambda}$ be the MLE for λ.

$$\text{Find} \qquad \lim_{n\to\infty} n\, E\Big(\hat{\lambda} - \lambda\Big)$$

(b). Let $Y_i \sim \text{LogNormal}(\mu, \sigma^2)$, $i = 1, \ldots, n$. Find an estimator $\exp(\tilde{\mu})$ of e^μ unbiased up to order $o(1/n)$ (ie such that $\lim_n E(\exp(\tilde{\mu}) - e^\mu) = 0$) based on $\{Y_i\}_{i=1}^n$ which asymptotically achieves the Cramer-Rao bound for variance of unbiased estimators. (*Extra: Why is it not possible to find an exactly unbiased estimator which for finite sample-size n exactly achieves the C-R bound* ?)

(IV) [*Confidence intervals, Pivotal Quantities*] Suppose that (X_i, Y_i) are *iid* as random pairs, $i = 1, 2, \ldots, n$, with $Y_i = \beta X_i + \epsilon_i \sqrt{X_i}$, where $X_i > 0$ are bounded variables and ϵ_i has mean 0, finite fourth moment, and is independent of X_i.

(a). Find a $1 - \alpha$ level confidence interval for β that is dual to the hypothesis test of $H_0 : \beta \le \beta_0$ which rejects when $\sum_{i=1}^n I_{[Y_i > \beta_0 X_i]} \ge B^{-1}(\cdot, n, 1/2)(1 - \alpha)$. (Right-hand side is $1 - \alpha$ quantile of $\text{Binom}(n, 1/2)$.)

(b). Show that, if $\beta^* = \beta_n^*$ is any consistent estimator (sequence) for β, then

$$\sum_{i=1}^n (Y_i - \beta X_i)/\sqrt{X_i} \Big/ \Big(\sum_{i=1}^n (Y_i - \beta^* X_i)^2 / X_i \Big)^{1/2}$$

is an 'asymptotically pivotal' quantity in the sense that its limiting distribution exists and does not depend on β or the probability law of ϵ_i. Use this pivotal quantity to create an asymptotically $(1 - \alpha)$-level confidence interval for β.

(c). Show that $(Y_i - \beta X_i)^2 / X_i$ is a contrast function $\rho((Y_i, X_i), \beta)$. Find the minimum contrast estimator and its asymptotic (large-n) distribution.