Sample Problems for Test 1

The following problems are, in topic and level of difficulty, like those which will be asked on the in-class test on Wednesday, March 20. The test will consist of four or five such problems, although eight are given here. (The 'extra' parts of the problems are mostly a little more difficult than the test problems will be.)

1. Suppose that $X_1, \ldots, X_n$ are iid r.v.'s with the density $(1/\vartheta)\, I_{[0,\vartheta]}$ for some $\vartheta > 0$. Find a most powerful test of significance level α for $H_0 : \vartheta = 1$ versus $H_A : \vartheta = 2$. Is it unique ? Is there some set of alternatives larger than $\{2\}$ for which your test is automatically most powerful ?

2. Show that the sufficient statistic T for an exponential-family random sample of size n is automatically monotone likelihood ratio if $w(\vartheta)$ is a strictly increasing function.

3. Suppose that $p_1(\mathbf{X})$ is the p-value for the UMP test of $\mathbf{H}_0 : \mu \le 0$ based on a sample of n iid $\mathcal{N}(\mu, 1)$ observations, and that $p_2(\mathbf{Y})$ is the p-value for the Likelihood Ratio Test of the same hypothesis based upon an independent sample of m iid $\mathcal{N}(\mu, \sigma^2)$ r.v.'s, where σ is unknown. What is the probability density of $\min(p_1(\mathbf{X}), p_2(\mathbf{Y}))$ when $\mu = 0$ for both the $\mathbf{X}$ and the $\mathbf{Y}$ samples ?

4. Let X_i be a sample of discrete, categorical r.v.'s, $1 \le 20$, with values A, B, C respectively occurring with probabilities $(4-\vartheta)/6$, $(1-\vartheta)/3$, and $\vartheta/2$, and with observed frequencies $n_A = 8$, $n_B = 6$, $n_C = 6$. Find the rejection region (in terms of n_A, n_B, n_C) for the most powerful test of size 0.05 of $H_0 : \vartheta = 0$ versus $H_A : \vartheta = 1$ *and find the p-value* for the given data values. You should use the following probability distribution function values for Binom$(20, 2/3)$:

$x =$	8	9	10	17	18
$B(x, 20, \frac{2}{3}) =$	.013	.037	.092	.940	.982

Note: Be careful on this problem: unlike other examples we have worked with, there is a data region where the H_0 probability is 0 while the alternative H_A probability is positive, and another region where the H_A probability is 0 while the H_0 probability is positive !

5. Consider the data X_i, $i = 1, \ldots, n$, with density of the form $f(x, \vartheta) = \vartheta x^{\vartheta - 1}$, $0 < x < 1$, where $\vartheta > 0$ is an unknown parameter. Find the form of the confidence set for ϑ obtained by inverting the family of Likelihood Ratio Tests for $H_0 : \vartheta = \vartheta_0$. Justify if you can why this set is an interval.

6. Suppose that (X_i, Y_i) are independent pairs of variables such that $X_i \sim \text{Expon}(\lambda)$ and conditionally given X_i, $Y_i \sim \mathcal{N}(\mu, 2/X_i)$. Find a pivotal quantity $Q(\mathbf{X}, \mathbf{Y}, \mu)$ for the parameter μ which has distribution not depending on λ.

Extra: if you can, find such a pivotal quantity which is a function of the joint sufficient statistics for (λ, μ).

7. Suppose that a hypothesis test of $H_0 : \vartheta = 0$ is based upon single observation $X \sim \text{Unif}[\vartheta, \vartheta + 2]$, $\vartheta \in [0, 1]$.
(a). Define and calculate the power function for the test which rejects H_0 whenever $X \geq 3/2$.
(b) Find the minimum-(Bayes)-risk procedure for the loss-function

$$L(\vartheta,\, a) \;=\; \begin{cases} 0 & \text{if} \quad a = Rej,\ \vartheta > 0 \\ 0 & \text{if} \quad a = Acc,\ \vartheta = 0 \\ 2 & \text{if} \quad a = Acc,\ \vartheta > 0 \\ 1 & \text{if} \quad a = Rej,\ \vartheta = 0 \end{cases}$$

with respect to the prior π which assigns probability-mass $1/2$ to each of $\vartheta = 0,\ 1$.

8. Assume that the large data-sample Y_i, $1 \leq i \leq n$, is distributed with density $2\lambda x e^{-\lambda x^2}$, $x > 0$, where $\lambda > 0$ is an unknown parameter. Find the rejection region for a UMP test of $H_0 :\ \lambda \leq 1$ with size approximately 0.95. You should specify the cutoff as well as the form of the rejection region. (Your answer may include some unevaluated Gamm-function values. but should otherwise be explicit.)