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Sample Problems for Test 1, Stat 701

(I). (a). Find an estimator of μ^2 which is unbiased up to $o(1/n)$ remainder, based on a size-n sample of $\mathcal{N}(\mu, \sigma^2)$ data. (b). Find such an estimator of the form $g_n(\bar{X}, S^2)$.
Note: there is a UMVUE in this problem which is exactly unbiased.

(II). (a). Find a UMP hypothesis test for $H_0 : \vartheta \le 1$ versus $H_A : \vartheta > 1$ based on data vector $\mathbf{X}_n$ (with dependent components) assumed to follow the density

$$f(\mathbf{x}, \vartheta) \;=\; \vartheta^n \, C_n / (1 + \vartheta \|\mathbf{x}\|)^{(n/2+1)}$$

where the constant C_n is chosen to make this a density, and does not depend on n. Here $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, and $\| \cdot \|$ is Euclidean distance.

(b). In the setting of part (a), show how to find a pivotal quantity, and how to use it to find a two-sided 95% confidence interval based on the n-component data $\mathbf{X}_n$.

(III). Show that $\bar{X}$ is **not** uniformly consistent for μ based on $\mathcal{N}(\mu, \sigma^2)$ data-samples, but **is** uniformly consistent if σ^2 is known (i.e., restricted) to lie in the interval $(0, 4]$.

(IV). Consider the Generalized Method of Moments estimator $\tilde{\lambda}$ of λ based on Expon(λ) data-sample $Y_1, \ldots, Y_n$ using the function $g(Y) = I[Y \le 1]$. What is the ARE (i.e., square of ratio of asymptotic CI widths) for this estimator versus the MLE $\hat{\lambda}$. Does the answer change if you are comparing the plug-in estimators of $\sqrt{\lambda}$ based on $\tilde{\lambda}$ versus $\hat{\lambda}$?

Additional problems of all four types are easy to make up, although most of type (II)(a) will be from Exponential families. I might also ask questions related to the existence of MLE's in exponential families, to the definition of contrast function and estimating function, to multivariate Information or Delta Method, and to the construction of Confidence Intervals using pivotal quantities or a test-based idea. Generalized Likelihood Ratio tests are also within scope for this exam.