Stat 701 Sample Problems for Test 2, Spring 2010

(I). (*Wald, Score and GLRT Tests*) A sample of observations $X_1, \ldots, X_n$ from a Gamma(α, β) density

$$f(x, \alpha, \beta) \;=\; \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}\, I_{[x>0]}$$

For all problem parts below, you may use the formula for the Fisher Information matrix (which we found in Problem (I) in HW4)

$$I(\alpha, \beta) \;=\; \begin{pmatrix} 1/\beta^2 & (.4228 - \ln\beta)/(\alpha\beta) \\ (.4228 - \ln\beta)/(\alpha\beta) & (.8237 - .8456\ln\beta + \ln^2\beta)/\alpha^2 \end{pmatrix}$$

You may express your answers in terms of I_{11}, I_{21}, I_{22} if you wish.

(a). Find the Rao Score test for $H_0 :\; \alpha = 1$ against the general (two-sided) alternative, including the approximate large-sample cutoff for your test statistic to give a size .05 test.

(b). Although there is no closed-form MLE in this problem, there is an easily derived equation from the β score equation which expresses $\hat{\alpha}$ uniquely in closed form in terms of $\hat{\beta}$. Find this expression.

(c). Use the expression in part (a) to give the Generalized Likelihood Ratio Test of size .05 (with large-sample approximate cutoff) in terms of $\hat{\beta}$ for $H_0 :\; \alpha = 1$ versus the general alternative.

(II). (*GMOM vs ML based estimators*) A sample $Z_1, \ldots, Z_n$ of $\mathcal{N}(\mu, \sigma^2)$ (with both parameters unknown) is observed, and it is desired to estimate a target parameter defined as the *signal-to-noise ratio* $\gamma = \mu/\sigma$ (which is the reciprocal of the more common *cofficient of variation*).

(a). Find the asymptotic variance of the substitution estimator for γ based on the Maximum Likelihood Estimators.

(b). Find the limiting distribution of the Generalized Method of Moments Estimator of γ based on $h(Z_i) = I_{[Z_i>0]}$.

(III). (*Univariate estimating equation*) Suppose you observe a data sample $W_1, \ldots, W_n$ from $\text{Expon}(\beta_0)$. Define an estimator $\tilde{\beta}$ as the solution to the estimating equation

$$\sum_{i=1}^{n} W_i \, (\beta \, W_i \; - \; 2) \; = \; 0$$

(a). Explain how you know that the estimator is consistent, and

(b). use what you know about estimating equations to find the asymptotic distribution of $\sqrt{n}\,(\tilde{\beta} - \beta_0)$.

(IV). (*Power of Test, in large-sample approx.*) (a). Find the approximate power versus $\vartheta = 1/3$ of the Most Powerful size 0.05 test of $H_0 : \vartheta = 1/4$ versus $H_A : \vartheta > 1/4$ based on $\text{Multinomial}(n, \; (\vartheta^2, \; 2\vartheta(1-\vartheta), \; (1-\vartheta)^2)$ data (n_0, n_1, n_2).

(b). Is your test in (a) UMP against $H_2 : \vartheta \geq 1/2$?

(V). (*Sample Size Calculation* Find the approximate sample size needed to test $H_0 : \; \lambda \leq 1.0$ versus $H_1 : \; \lambda > 1.0$ at significance level $\alpha = 0.01$ and achieve power 0.90 at $\lambda = 1.5$, based on a $\text{Poisson}(\lambda)$ data sample $Y_1, \ldots, Y_n$. (Note: $\Phi^{-1}(.995) = 2.576$.)