In-Class Test, Stat 701, 9-9:55am, 3/17/23

The test is closed-book, but you may use a single-sided notebook sheet of formulas as a memory aid. Reduced numerical answers are not required in any problem, and you may leave quantities $z_\gamma = \Phi^{-1}(1-\gamma)$ as symbols rather than numbers in your answers.

(1). (*36 pts.*) A sample of n observations $X_1, \ldots . X_n$ is drawn from the density $f(x, \theta) = 2\theta x \, \exp(-\theta \, x^2) \, I_{[x>0]}$, where θ is unknown.

(a). Find the UMP test of $H_0 : \;\theta \le \theta_0$ versus $H_A : \;\theta > \theta_0$ with its large-sample approximate rejection threshold at significance level $\alpha = 0.05$.

(b). Find a test-based one-sided Confidence Interval of approximate confidence level 0.95 for θ based on the data and the family of hypothesis tests in (a).

(c). Find a test with significance level 0.05 of $H_0 : \theta \le \theta_0$ versus $H_A : \theta > \theta_0$ based on the Maximum Likelihood test statistic $\hat{\theta}$ using its large-sample approximate distribution.

(2). (*36 pts.*) Based on the MLE $\hat{\lambda}$ for λ using a Poisson(λ) data-sample $Y_1, \ldots, Y_n$:

(a). Give a large-sample test for $H_0 : \lambda \le 2$ versus $H_1 : \lambda > 2$, and find its approximate power at $\lambda = 2 + b/\sqrt{n}$ for fixed $b > 0$.

(b). Use your answer in (a) to find approximately how large a sample-size n is needed so that a test of size $\alpha = 0.05$ based on $\hat{\lambda}$ for a Poisson(λ) sample of size n will achieve 90% power against the alternative $\lambda = 2.25$.

(3). (*34 pts.*) A sample of *iid* data $X_1, \ldots, X_n$ is assumed to come from the density

$$f(x, \mu) \;=\; 2\,(1 + |x - \mu|)^{-5} \quad , \qquad -\infty < x < \infty$$

for some value μ_0 of the unknown positive parameter μ.

(a) An estimator of μ is this problem is defined by

$$n^{-1} \sum_{i=1}^{n} I_{[X_i \le 0]} \;=\; F(0, \tilde{\mu})$$

where $F(x, \mu)$ is the distribution function associated with the density $f(x, \mu)$. Find the function $F(0, \mu)$ for positive μ. Explain why the root $\tilde{\mu}$ will be positive with high probability when $\mu_0 > 0$ and n is large.

(b) Find the asymptotic variance of the estimator $\tilde{\mu}$.

Hint: This density is symmetric about μ, and is differentiable everywhere except at $x = \mu$.