Major Topics & Problem Types for In-Class Test, 3/10/2025

The in-class test on Friday, March 17, 2023, will consist of three problems. The test is closed-book, but you may bring a single-sided formula sheet as a memory aid if you like.

The major topics that we covered that may figure on the test are:

(1) Duality between Confidence Intervals (1- or 2-sided) and Hypothesis Tests (§4.5)

(A). Likelihood Ratio Tests and their dual Confidence Intervals.

(B). Bayesian Confidence Interval analogues: credible sets

(2) Approximate Large-Sample Pivots and approximate Confidence Intervals and Tests with endpoints determined using the Central Limit Theorem

(A). Delta Method and variance Stabilizing Transformations (§5.3)

(B). Large sample theory of MLEs for Canonical Exponential Families (§5.3.3)

(C). Approximate tests and confidence intervals based on Generalized Method of Moments

(3) Large-sample theory of Maximum Likelihood Estimators and Estimating Equation Estimators (§5.3 − −5.4) for One-dimensional Parameters

(A). Sufficient Conditions for Consistency of Minimum Contrast Estimators

(B). Examples of Minimum Contrast and Estimating Equation Estimators

(C). Optimality of MLE among estimating-equation estimators. Relative Efficiency.

(D). Approximate Power of Large-Sample Wald Tests versus Contiguous Alternatives.

(4) Large-sample Bayes theory of Maximum Likelihood Estimators and Estimating Equation Estimators (§5.4)

See next page for Sample Problems.

Sample Problems for In-Class Stat 701 Test

(1). *too hard* Show using the method of problem 5.4.1 that the solution of the estimating equation $\sum_{i=1}^{n} \psi(X_i - \theta)$ for $\psi(x) = 0.2\, I_{[x>0]} - 0.3\, I_{[x<0]}$ is a consistent and asymptotically normal estimator of the 0.4 quantile ξ of the distribition function F of X_i, under the assumption that F is continuous and ξ is a point of increase of F and $F'(\xi) = f(\xi)$ exists, and find the asymptotic variance of the estimating-equation solution given by the sample 0.4 quantile.

(2). Suppose that, based on a sample of $n = 100$ observations $Y_i \sim f(y, \theta)$, you are interested in hypothesis tests about the value of θ in the neighborhood of $\theta_0 = 1$. Suppose that the density f satisfies all of the standard MLE-theory regularity conditions, and that the score statistic $S_n = n^{-1/2} \sum_{i=1}^{100} \nabla_\theta \log f(Y_i, \theta_0) = 3.7$ and

$$n^{-1} \sum_{i=1}^{100} \left[\nabla_\theta \log f(Y_i, \theta_0)\right]^2 \;=\; 2.56$$

(a). Find the approximate value of the MLE $\hat{\theta}$ based on the data $(Y_1, \ldots, Y_n)$.

(b). What is the approximate one-sided p-value for a hypothesis test of $H_0 : \theta = \theta_0$ versus $H_A : \theta > \theta_0$ based on these data ? (Define your test using an asymptotic pivot based on $\hat{\theta}$.)

(3). A large sample of n observations is drawn from the density $f(x, \theta) = 4\theta/(1 + \theta x)^5$ for $x > 0$, where $\theta > 0$ is an unknown parameter. Find the asymptotic relative efficiency of the method of moments estimator versus the MLE. You do not need to find the explicit form of the MLE to solve this problem, but you should find the estimating equation that it solves and verify that the MLE is unique. *Note: to solve this problem, you may use the identity*

$$\int_0^\infty \theta^{k+1}\, x^k\, (1 + \theta\, x)^{-m}\, dx \;=\; k!\,(m-k-2)!/(m-1)! \;, \qquad m \geq k+2$$

(4). An *iid* sequence of $\text{Geom}(p)$ random variables $M_1, M_2, \ldots, M_{40}$ is observed ($p_M(k) = (1-p)^{k-1}\, p$ and $EM = 1/p$, $\text{Var}(M) = (1-p)/p^2$, with $T = \sum_{j=1}^{40} M_j = 87$.

(a). If the unknown parameter p follows a $\text{Beta}(0.5, 0.5)$ distribution, then find a 95% two-sided equal-tailed credible interval for p.

(b). Find an approximate frequentist CI for p based on the same data.

Additional types of problems: Delta-Method of first or 2nd order; Exponential-family MLE and Information and (large-sample) Confidence Interval; Approximate power for large-sample hypothesis test of θ_0 *based on estimating-equation estimator* $\tilde{\theta}$ *against alternative* $\theta = \theta_0 + c/\sqrt{n}$.