APPENDIX II.

Pointwise convergence in probability of random concave functions implies uniform convergence on compact subspaces.

The "almost sure" version of this theorem is a direct consequence of Rockafellar (1970, Theorem 10.8). However for an "in probability" result we must be more careful. The following "diagonalization method" was pointed out by T. Brown.

THEOREM II.1. *Let E be an open convex subset of $\mathbb{R}^p$ and let $F_1, F_2, \cdots$, be a sequence of random concave functions on E such that $\forall x \in E$, $F_n(x) \to_{\mathscr{P}} f(x)$ as $n \to \infty$ where f is some real function on E. Then f is also concave and for all compact $A \subset E$,*

$$\sup_{x\in A} | F_n(x) - f(x)| \to_{\mathscr{P}} 0 \text{ as } n \to \infty.$$

PROOF. Concavity of f is obvious. Next let $x_1, x_2 \cdots$ be a countable dense set of points in E. Since $F_n(x_1) \to_{\mathscr{P}} f(x_1)$ as $n \to \infty$ there exists a subsequence along which convergence holds almost surely. Along this subsequence $F_n(x_2) \to_{\mathscr{P}} f(x_2)$ so a further subsubsequence exists along which also $F_n(x_2) \to_{\text{a.s.}} f(x_2)$. Repeating the argument, along a $(\text{sub})^k$ sequence, $F_n(x_j) \to_{\text{a.s.}} f(x_j)$ for $j = 1, \cdots, k$. Now consider the new subsequence formed by taking the first element of the first subsequence, the second of the second, etc. Along the new subsequence we must have $F_n(x_j) \to_{\text{a.s.}} f(x_j)$ for each $j = 1, 2, \cdots$.

By Rockafellar (1970, Theorem 10.8) it now follows that

$$\sup_{x\in A} | F_n(x) - f(x)| \to_{\text{a.s.}} 0 \text{ along this subsequence.}$$

We have shown more generally how, from any subsequence, a further subsequence can be extracted along which $\sup_{x\in A} | F_n(x) - f(x)| \to_{\text{a.s.}} 0$. It now follows that

$$\sup_{x\in A} | F_n(x) - f(x)| \to_{\mathscr{P}} 0 \text{ along the whole sequence.} \quad \square$$

COROLLARY II.2. *Suppose f has a unique maximum at $\hat{x} \in E$. Let $\hat{X}_n$ maximize F_n. Then $\hat{X}_n \to_{\mathscr{P}} \hat{x}$ as $n \to \infty$.*

PROOF. The proof, a simple $\varepsilon - \delta$ argument, is left to the reader. $\square$

APPENDIX III.

Extension of SLLN for $D[0, 1]$.

Let $X; X_1, X_2, \cdots$ be i.i.d. random elements of $D[0, 1]$ with $\mathscr{E}\|X\| = \mathscr{E} \sup_{t\in[0,1]} |X(t)| < \infty$. Then by Theorem 1 of R. Ranga Rao (1963) we have almost surely

$$\left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathscr{E}X \right\| \to 0 \quad \text{as } n \to \infty.$$

We need to extend this result in two directions. Firstly we must allow the random elements of $D[0, 1]$ to be random functions not from $[0, 1]$ to $\mathbb{R}$ but from $[0, 1]$ to the space of continuous real functions on $\mathscr{B}$, where $\mathscr{B}$ is a compact neighbourhood of $\beta_0 \in \mathbb{R}^p$. If we endow this space of functions with the supremum norm it becomes a separable Banach space, and that will be all the structure we need.

Secondly we must allow for censoring. To tie in with the usual right continuity convention for $D[0, 1]$, we shall consider *left* censoring: X_i is only observed on an interval $[t_i, 1]$, or more generally, in a triangular array scheme, on $[t_i^{(n)}, 1]$ or $[T_i^{(n)}, 1]$ for fixed or random times $t_i^{(n)}$ or $T_i^{(n)}$ respectively.

THEOREM III.1. *Let $X; X_1, X_2, \cdots$ be* i.i.d. *random elements of $D_E[0, 1]$ (endowed with the Skorohod topology) where the elements of $D_E[0, 1]$ are right continuous functions on $[0, 1]$ with left hand limits taking values in a separable Banach space E (rather than the usual $\mathbb{R}$). Suppose that $\mathscr{E}\|X\| = \mathscr{E} \sup_{t\in[0,1]} \|X(t)\| < \infty$.*