Stat 710 HW1 Solutions

(2). The density of the t_n distributed random variable $X_n = Z_n/\sqrt{V_n}$ has density of the form $C_n\,(1+x^2/n)^{-(n+1)/2}$, where $Z_n \sim \mathcal{N}(0,1)$ and $V_n \sim \mathit{Gamma}(n/2, n/2)$ are independent. We know that the distributional convergence of X_n to $\mathcal{N}(0,1)$ alone is not enough to ensure convergence of moments: we need some uniform integrability. Moreover, it is easily checked that the p^{th} moment of X_n does not even exist for $n \le p$. However, that is no obstacle to the p^{th} moment convergence of X_n to $\mathcal{N}(0,1)$, since such convergence relates only to the behavior of integrals *for sufficiently large* n for each fixed p, and does in fact hold. To check the sufficient condition

$$\{\, X_n^p : \quad n \ge 2p+2\} \qquad \text{is uniformly integrable}$$

it is enough to note that $E|Z_n|^{p+1}$ and $E|V_n|^{-(p+1)/2}$ are uniformly bounded for values of $n \ge 2p+2$, and the latter can easily be checked using the integrals with respect to *Gamma* densities.

(6). This problem was done in class, two ways.

(10). If $X_n \longrightarrow 0$ in probability and $\{Y_n\}_{n\ge1}$ is tight, then for all $\epsilon > 0$, M, n_0 can be found so large that for all $n \ge n_0$,

$$P(|X_n| \ge 1) \le \frac{\epsilon}{2}\,, \quad P(|Y_n| \ge M-1) \le \frac{\epsilon}{2} \qquad \Longrightarrow \quad P(|X_n+Y_n| \ge M) \le \epsilon$$

which overall says that $X_n + Y_n = \mathcal{O}_P(1)$.

(12). (i) Take $R(h) = h^2$, and either $X_n \sim \text{Unif}[1,2]$ or $X_n = n,\ 1/n$ each with probability $1/2$. In either case, $R(X_n)/X_n = X_n$ does not conerge in probability to 0.
(ii) Take $R(h) = h/(1-h)$, or $h \log\frac{1}{1-h}$, and $X_n \sim \text{Unif}(1-1/n, 1)$. Then $R(X_n)/X_n$ actually converges to ∞ in probability as $n \to \infty$, in the sense that $1/X_n$ converges in probability to 0.

(16). If A is an orthogonal $n \times n$ matrix, and $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, I_n)$, then we know by Jacobian change of variable formulas that $A\,\mathbf{X} \sim \mathcal{N}(\mathbf{0}, I_n)$, and by linear algebra $\|\mathbf{X}\|^2 = \|A\,\mathbf{X}\|^2$. Therefore $A\,\mathbf{U} = A\,\mathbf{X}/\|\mathbf{X}\| = A\,\mathbf{X}/\|A\,\mathbf{X}\|$ has the same distribution as $\mathbf{U}$, supported on $\mathbf{S}^{n-1} = \{\mathbf{u} \in \mathbf{R}^n : \ \|\mathbf{u}\| = 1\,\}$.

(17). By the Law of Large Numbers, $\|\mathbf{X}\|^2/n \longrightarrow 1$ in probability as $n \to \infty$, so that

$$\sqrt{n}\,(U_{n,1},\, U_{n,2}) \;=\; \sqrt{n/\|\mathbf{X}\|}\,(X_1,\, X_2) \;=\; (X_1,\, X_2) \,+\, o_P(1)$$

(18). For $\delta > 0$ chosen arbitrarily small, choose a continuity point M of the limiting distribution of $\sqrt{n}\,|T_n - \theta|$ which is larger than the $1-\delta$ quantile of that limiting distribution. Then by construction,

$$\limsup_n \; P(|T_n - \theta| \ge \delta) \;\le\; \limsup_n \; P(|T_n - \theta| \ge M/\sqrt{n}) \;\le \delta$$

(1). [**Corrected Solution**] To make the multivariate Central Limit Theorem apply, you need to express S^2 asymptotically equivalent to an empirical average (of a function of the *iid* variables X_i). In fact,

$$\sqrt{n}\,(S^2 - \sigma^2) \;=\; \sqrt{n}\,\left(\frac{1}{n-1}\Big[\sum_{i=1}^n X_i^2 \;-\; n\overline{X}^2\Big] \;-\; \sigma^2\right)$$

$$=\;\; \frac{\sqrt{n}}{n-1}\left(\sum_{i=1}^n (X_i^2 - \mu^2 - \sigma^2) \;-n\Big((\overline{X} - \mu + \mu)^2 - \mu^2\Big) \,+\, \sigma^2\right)$$

$$=\; \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^2 - \mu^2 - \sigma^2 - 2\mu(X_i - \mu)) \,+\, o_P(1)$$

from which we easily conclude that the CLT applies to $\sqrt{n}\left(\overline{X} - \mu,\, S^2 - \sigma^2\right)$ and that the two coordinate variables are asymptotically independent if their asymptotic covariance is 0, i.e., if (in terms of $EX_1^3 = \mu_3$)

$$E\Big((X_1 - \mu)\,(X_1^2 - \mu^2 - \sigma^2 - 2\mu(X_1 - \mu))\Big) \;=\; \mu_3 - \mu(\mu^2 + \sigma^2) - 2\mu\sigma^2 \;=\; 0$$

(7). Since for smooth monotone g, $n^{-1/2} \sum_{i=1}^n \big(g(X_i) - g(\theta)\big) \sim \mathcal{N}(0, \theta\,(g'(\theta))^2)$, we have variance stabilization if $g(\theta) = \sqrt{\theta}$, and the confidence interval becomes $\Big(n^{-1} \sum_{i=1}^n \sqrt{X_i} \;\pm\; z_{\alpha/2}/(2\sqrt{n})\Big)^2$.