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Solutions to Stat 710 Problem Set 3

#6.1. Here we make explicit use of the likelihood ratio $\exp(W_n) \equiv (dQ_n/dP_n)$ $= \exp(-\mu_n^2/2 + X\,\mu_n)$, where $X \sim \mathcal{N}(0,1)$ under P_n. We know tightness of e^{W_n} under P_n which implies that every infinite sequence of integers n has a subsequence under which $\exp(W_n)$ must converge in P_n distribution, and all corresponding limits of W_n must be normal (easy to prove using characteristic functions) $\mathcal{N}(-\mu_*^2/2,\, \mu_*^2)$ where $\mu_n \to \mu_* < \infty$ along the subsequence. (So far, μ_* might be $-\infty$.) Now apply the criteria in the First Lecam Lemma (Lemma 6.4) directly. For contiguity of Q_n to P_n, (iii) says that all possible limits μ_* must be finite (i.e., not $-\infty$). For contiguity of P_n to Q_n, (ii) with roles of P_n and Q_n reversed says also that all such limit points must be finite. Thus the necessary and sufficient condition for either or both directions of contiguity is that $\{\mu_n\}_n$ be a bounded sequence.

#6.2. Since the mean $\bar{X}$ of a sample n from the $\mathcal{N}(\vartheta, 1)$ is $\mathcal{N}(\vartheta, n^{-1})$, this problem is like the last one except that the likelihood ratio $\exp(W_n) = \exp(-n\vartheta_n^2/2 + n\,\vartheta_n\,\bar{X})$. Again it is easy to see that the only possible P_n distributional limits of $\exp(W_n)$ are $\log(\mathcal{N}(-\alpha_*,\, \alpha_*))$, where α_* is a limit point (nonnegative but possbly infinite) of $n\,\vartheta_n^2$. Using exactly the same criteria as before for one or both directions of contiguity of P_n and Q_n we find that any of these condition of contiguity is equivalent to boundedness of $n\vartheta_n^2$.

#6.3. First consider the case where P_n and Q_n are laws of a single Unif$[0,1]$ and Unif$[0, 1+n^{-1}]$ coordinate (not the situation asked in the Exercise.) Here dQ_n/dP_n is precisely $\frac{n}{n+1}\, I_{[0,1]}$, which evidently satisfies (ii) of Lemma 6.4 with roles of P_n, Q_n reversed. Moreover, since $Q_n([0,1]) \to 1$, criterion (iii) is also satisfied. Thus P_n and Q_n are mutually contiguous.

Now in the situation of the Exercise, P_n and Q_n are respectively the laws of n *iid* coordinates, Unif$[0,1]$ and Unif$[0, 1+n^{-1}]$. Then $dQ_n/dP_n = (\frac{n}{n+1})^n\, I_{[0,1]^n} \to e^{-1} I_{[0,1]^n}$ in P_n distribution. The limit is a.s.> 0, so P_n is contiguous to Q_n by Lemma 6.4(ii) with P_n, Q_n reversed. But the limit does not have expectation 1, so by Lemma 6.4(iii) Q_n is not contiguous to P_n.

#6.4. Assume $\|P_n - Q_n\| \to 0$. By definition, for any sequence of measurable events $|P(A_n) - Q(A_n)| \to 0$, so that $P(A_n) \to 0$ if and only if $Q(A_n) \to 0$. Thus, again by definition, each of P_n, Q_n is contiguous with respect to the other.

#6.6. Simple examples departing from the one in #6.3 are readily constructed. If Q_n is Lebesgue measure on $[0,1]$ and P_n is Lebesgue measure on any interval containing $[0,1]$, of length b_n such that $\liminf_n b_n > 1$ and $\limsup_n b_n < \infty$, then $dQ_n/dP_n = (1/b_n)\, I_{[0,1]}$ and criterion (ii) with P_n, Q_n reversed holds, but criterion (iii) does not. Thus Q_n is contiguous to P_n, but P_n is not contiguous to Q_n.