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Partial Solutions for STAT 730 HW1, Spring 2026

Problem 1. First express the condition that the random variables X_i, X_j are conditionally independent given the random vector $W \equiv \{X_a : 1 \le a \le K, a \ne i, a \ne j\}$. The simplest way is to say that the variables X_i, X_j and W are jointly Gaussian and that the regression of X_i on X_j, W has coefficient 0 for the regression coefficient on X_j. Consider

$$\begin{pmatrix} X_i \\ X_j \\ W \end{pmatrix} \sim \mathcal{N}_K\big(\underline{0}, V\big) \ , \qquad V \equiv \begin{pmatrix} a & v^{tr} \\ v & C \end{pmatrix} \quad \text{nonsingular}$$

where a is scalar, $v \in \mathbb{R}^{K-1}$, and C is $(K-1) \times (K-1)$ nonsingular. For these jointly Gaussian variables

$$X_i - v^{tr} C^{-1} \begin{pmatrix} X_j \\ W \end{pmatrix} \qquad \text{is independent of} \quad (X_j, W)$$

Therefore the regression coefficient of X_i on X_j in the regression (linear projection of X_i onto the span of (X_j, W)) is $(C^{-1}v)_1$, i.e, the first element of $C^{-1}v$. So our task is to show that $C^{-1}v = 0$ if and only if $(V^{-1})_{12} = 0$.

To find the expression for V^{-1}, we solve the equations $V \binom{x}{y} = \binom{\xi}{\eta}$, where $y, \eta \in \mathbb{R}^{K-1}$. These equations determine elements of V^{-1} such that $V^{-1}\binom{\xi}{\eta} = \binom{x}{y}$. The equations can be written in the form

$$ax + v^{tr}y = \xi \ , \qquad x\,v + Cy = \eta$$

Taking the first equation minus $v^{tr}C^{-1}$ left-multiplied by the second gives $(a - v^{tr}C^{-1}v)\,x = \xi - v^{tr}C^{-1}\eta$, which expresses in terms of the first row $(V^{-1})_{(1)}$ of V^{-1}

$$x = (V^{-1})_{(1)} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = (a - v^{tr}C^{-1}v)^{-1}\,(\xi - v^{tr}C^{-1}\eta)$$

This last equation implies that $(V^{-1})_{12}$ is the coefficient of the first element of η in $(a - v^{tr}C^{-1}v)^{-1}(\xi - v^{tr}C^{-1}\eta)$, or equivalently, the first element of the row vector $-(a - v^{tr}C^{-1}v)^{-1}\,v^{tr}C^{-1}$. But this says

$$(V^{-1})_{12} = -(a - v^{tr}C^{-1}v)^{-1}\,(C^{-1}\,v^{tr})_1$$

is 0 if and only if $(C^{-1}\,v^{tr})_1 = 0$, which was to be shown.

Problem 2. (a) Clearly $E(Y_t) = E(Z_t) = 0$ for all t. A trignometric identity for $cos(a-b)$ shows that

$$E(Y_t\, Y_{t+h}) = \sigma^2 \left\{ \sin(2\pi \frac{t}{\omega_0}) \sin(2\pi \frac{t+h}{\omega_0}) + \cos(2\pi \frac{t}{\omega_0}) \cos(2\pi \frac{t+h}{\omega_0}) \right\} = \sigma^2 \cos(2\pi \frac{h}{\omega_0})$$

is free of t, establishing wide-sense stationarity of Y_t. Next, with $\eta \sim$ Unif$(0,\pi)$,

$$E(Z_t\, Z_{t+h}) = \frac{\sigma^2}{\pi} \int_0^\pi \left\{ \sin(\frac{2\pi t}{\omega_0}) \cos(\eta) + \cos(\frac{2\pi t}{\omega_0}) \sin(\eta) \right\} \cdot$$
$$\left\{ \sin(2\pi \frac{t+h}{\omega_0}) \cos(\eta) + \cos(2\pi \frac{t+h}{\omega_0}) \sin(\eta) \right\} d\eta$$

Since it is easy to check that $\int_0^\pi \sin(x)\cos(x)\, dx = 0$, $\int_0^\pi \cos^2(x)\, dx = \int_0^\pi \sin^2(x)\, dx = \pi/2$, it follows that

$$E(Z_t\, Z_{t+h}) = \frac{\sigma^2}{2} \left[\sin(2\pi \frac{t}{\omega_0}) \sin(2\pi \frac{t+h}{\omega_0}) + \cos(2\pi \frac{t}{\omega_0}) \cos(2\pi \frac{t+h}{\omega_0}) \right] = \frac{\sigma^2}{2} \cos(2\pi \frac{h}{\omega_0})$$

Since this is free of t, we have shown that Z_t is also wide-sense stationary.

We do parts (b) and (c) for Y_t next. It is easy to see that all Y_t variables jointly are Gaussian, because they are linear combinations of the same two independent Gaussian random variables ξ_1, ξ_2. Since we already remarked in class that a Gaussian wide-sense-stationary random sequence is automatically strictly stationary, claim (b) for Y_t is established. But the same claim for Y_t defined in terms of double-exponential ξ_i is false, because we can explicitly calculate the characteristic function of Y_t in the double-exponential case. The characteristic function of double-exponential is

$$\frac{1}{2} \int_{-\infty}^{\infty} e^{ix\lambda - |x|}\, dx = \frac{1}{2(1-i\lambda)} + \frac{1}{2} \int_{-\infty}^{0} e^{x(1+i\lambda)}\, dfx = \frac{1}{1+\lambda^2}$$

and from this it follows that

$$E(e^{i\lambda Y_t}) = \frac{1}{1+\lambda^2 \sin^2(2\pi t/\omega_0)} + \frac{1}{1+\lambda^2 \cos^2(2\pi t/\omega_0)}$$
$$= \frac{2(1+\lambda^2)}{(1+\lambda^2 \sin^2(2\pi t/\omega_0)) \cdot (1+\lambda^2 \cos^2(2\pi t/\omega_0))}$$

which is not constant as a function of integers t unless $2/\omega_0$ is an integer multiple of $1/2$.

The second part of (b) is not stated quite right. The process $Z_t = \xi \sin(2\pi t/\omega_0 + \eta)$ is strictly stationary if $f_\xi(x) = \frac{1}{2} |x|\, e^{-x^2/2}$ (a *double-Rayleigh* density) but *not* if $\xi \sim \mathcal{N}(0,1)$. The point is that

$$Z_t = (\xi \cos(\eta)) \sin(\frac{2\pi t}{\omega_0}) + (\xi \sin(\eta)) \cos(\frac{2\pi t}{\omega_0})$$

So Z_t is just like Gaussian Y_t if $(\xi \sin(\eta),\ \xi \cos(\eta))$ has the distribution of two independent Gaussian variables, and a Jacobian change-of-variables shows that is true when ξ is double-Rayleigh but not when it is Gaussian (and $\eta \sim \text{Unif}(0, \pi)$ is independent of ξ).

Problem 4. The Poisson process $N(t) - N(0)$ is not wide-sense stationary because it has mean λt and variance λt, but the series $N(t) - N(t-1)$ for $t \geq 1$ is *iid* with Poisson(λ) distribution and is therefore strictly stationary. It follows that $N(t)$ is a homogeneous-transition Markov sequence and so is $Z(t)$, where for $a = 0, 1$

$$P(Z(t{+}1) \;=\; 1{-}a \,|\, Z(t) = a) \;=\; \sum_{j=0}^{\infty} e^{-\lambda}\, \frac{\lambda^{2j+1}}{(2j+1)!} \;\equiv\; q(\lambda) \;=\; e^{-\lambda}\, \frac{1}{2}\, (e^{\lambda}{-}1) \;=\; (1{-}e^{-\lambda})/2$$

and by induction it follows that if $P(Z_t = a) = 1/2$ for $a = 0, 1$ (assumed true if $t = 0$), then

$$P(Z_{t+1} = b) \;=\; P(Z_t = 1 - b)q(\lambda) \;+\; P(Z_t = b)(1 - q(\lambda)) \;=\; 1/2$$

From this, we see that the Markov chain Z_t is in its equilibrium (stationary) distribution, and using the Markov property and the time-homogeneous transitions, for all K the joint probability mass function of $(Z_t, Z_{t+1}, \ldots, Z_{t+K})$ is the same for all t, proving that the $\{0, 1\}$ valued time series Z_t is strictly stationary.