April 10, 2017 5pm – 6:20pm

In-Class Test, STAT 730

Instructions: This test is closed-book but open-note. You may use a two-sided 8.5″ × 11″ notebook sheet for reference during the test, but no books or other reference materials. You may use a calculator. For full credit, you must justify all your steps and assertions by making explicit reference wherever possible to theorems covered in the book or in class, and by stating explicitly why the hypotheses of those theorems apply to the problem situations. The test consists of 3 problems, with point values as indicated. The maximum score on the test is 100 points, and you may answer any combination of problem sub-parts.

(1). (*35 pts.*) Suppose that X_t for $t = 0, \pm 1, \pm 2, \ldots$, is a stationary ARMA$(p, q)$ process satisfying $\phi(B)\, X_t = \theta(B)\, W_t$ for all t, for a square-integrable white-noise pocess $W_i \sim \mathrm{WN}(0, \sigma^2)$, where $\phi(z)$, $\theta(z)$ are polynomials with no common roots and where $\phi(z)$ has no complex roots z for which $|z| \le 1$. Let $\gamma(h)$ denote the autocovariance function of X_t. Show that:

(a) $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, and assuming (a), show that

(b) All roots z of $\theta(z)$ have $|z| \ne 1$ if and only if

$$\text{for all} \quad \lambda \in (-\pi, \pi], \qquad \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{ih\lambda} > 0$$

(2). (*50 pts.*) Let X_t be the stationary time series solution X_t of the equation

$$X_t - (1/2)\, X_{t-1} = W_t - (1/3)\, W_{t-1} \quad , \qquad W_t \sim \mathrm{WN}(0, \sigma^2)$$

(a) Find $\gamma(0), \gamma(1)$, where $\gamma(h)$ is the autocovariance of X_t at lag h.

(b) Find a general expression for $\gamma(h), \;\; h > 1$.

(c) Find an explicit expression $W_t = \sum_{j=0}^{\infty} \pi_j\, X_{t-j}$, with $\pi_0 = 1$.

(d) Find $E(X_t - \hat{X}_t)^2 \equiv v_{t-1}$, the mean-squared error of the best linear predictor of X_t in terms of $(X_s, \; s < t)$.

(e) Use (c) twice (once for X_t and once for X_{t-1}) to find the best linear predictor for X_t in terms of $(X_s, \; s \le t - 2)$.

(3). (*35 points*) Find the asymptotic variance V for $\sqrt{n}\, (\hat{\gamma}(1) - \gamma(1))$ based on observations $X_1, \ldots, X_n$ from the MA(1) time series $X_t = W_t + \beta\, W_{t-1}$, if $0 < \beta < 1$ (with β unknown) and the white noise sequence W_t is normally distributed with mean 0 and variance 1. Use this result to give an approximate large-sample confidence interval for $\gamma(1)$ based on the estimates $(\hat{\gamma}(0), \; \hat{\gamma}(1)) = (1.2, \; 0.55)$ from $X_1, \ldots, X_{100}$, with $n = 100$.
(*You need not simplify your arithmetic expression for the confidence interval.*)