April 2, 2017

Sample Problems for In-Class Test, STAT 730

(1). Consider the following difference-equations connecting a time series $\{X_t\}_{t=-\infty}^{\infty}$ with an associated White-noise sequence $\{W_t\}_{t=-\infty}^{\infty}$. Which of them have stationary solutions X_t ? For each, justify stationarity or explain why it does not hold.

(a) $X_t + X_{t-1} + X_{t-2} \;=\; W_t$

(b) $X_t - 0.8 X_{t-1} \;=\; 2W_t + W_{t-1}$

(c) $X_t + X_{t-2} \;=\; W_t$

(2). For each series in (1) that you identified as stationary: is the series causal ? invertible ?

(3). For the stationary time series solution X_t of the equation

$$X_t = W_t + W_{t-1} + 0.25 W_{t-2} \quad , \qquad W_t \sim WN(0, \sigma^2)$$

(a) Find the best linear predictor of X_3 in terms of $(X_s,\ s = 2, 1, 0, -1, \ldots)$, and find the mean squared prediction error.

(b) Find a general expression for W_t in terms of $(X_s,\ s \le t)$.

(4). If $X_t = W_t - 1.5W_{t-1} + 0.5W_{t-2}$ for all $t \ge 1$, where $(W_s,\ \ s \ge -1)$ is a White Noise with mean 0 and variance 1, then find the best linear predictor for X_3 in terms of X_1, X_2. Also find (but do not reduce) a numerical expression for the mean-squared error of this predictor.

(5). Consider the ARMA(1,1) stationary process X_t defined in terms of a White Noise $(0, \sigma^2)$ process W_t by the equation $X_t - aX_{t-1} = W_t + bW_{t-1}$, where $|a|, |b| < 1$. Then find $\text{Cov}(X_t, W_t)$, $\text{Cov}(X_t, W_{t-1})$, $\gamma_X(0)$, and $\gamma_X(1)$.

(6). Find the spectral density of the proces X_t in (5).

(7). Is a stationary causal linear time series always invertible ? Prove or give a counterexample.

(8). Suppose that $X_t = \alpha X_{t-1} + \beta X_{t-2} + W_t$ is a causal AR(2) process, where W_t is a $(0,1)$ White Noise sequence. Suppose for a large n that the column $(X_3, \ldots, X_n)^{tr}$ is linearly regressed on the $(n-2) \times 3$ design matrix $\mathbf{X}$ with first column all 1's, second column $(X_2, \ldots, X_{n-1})^{tr}$ and third column $(X_1, \ldots, X_{n-2})^{tr}$. Denote the least-squares regression coefficients by $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2$. What will be the limiting values a_{j*}, $j = 0, 1, 2$, for these three fitted coefficients ? Express the large-n limiting distribution of $\sqrt{n}\,(\hat{a}_j - a_{j*})_{j=0,1,2}$ as explicitly as you can in terms of α, β.

(9). Suppose that $X_t = A_1 \cos(\lambda t) + A_2 \sin(\lambda t) + W_t$ for all integers t, where A_1, A_2 are independent random variables with mean 0 and variance σ_A^2 and W_t is a $\mathrm{WN}(0, \sigma^2)$ sequence independent of (A_1, A_2). Show that X_t is wide-sense stationary but not strictly stationary. Also find its autocovariance function and spectral distribution function.
Hint: the spectral d.f. has jumps, so there is no spectral density.

(10). Find the asymptotic variance-covariance function V for the vector

$$\sqrt{n}\,\Big(\hat{\gamma}(0) - \gamma(0),\ \hat{\gamma}(1) - \gamma(1),\ \hat{\gamma}(2) - \gamma(2)\Big)$$

based on observations $X_1, \ldots, X_n$ from the time series X_t in problem (3) above, if the white noise sequence W_t is normally distributed. Explain how you would use this result to form a confidence interval for $\gamma(1)$ based on the estimates $(\hat{\gamma}(h),\ \ h = 0, 1, 2)$.