Pedestrian Flow Models with Slowdown Interactions

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Outline

Goal: Systematic Derivation of PDE Models for Pedestrian Traffic Flow

- Microscopic Rules for the Interaction of Pedestrians Moving in Opposite Directions

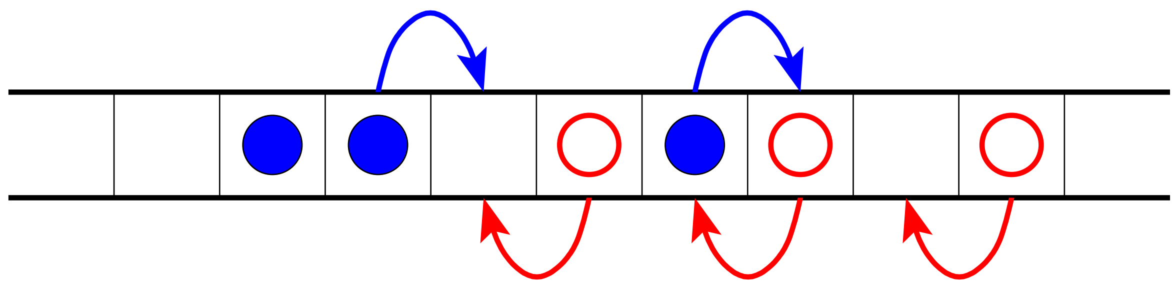
- Microscopic Cellular Automata Model for Pedestrian Flow

- Derivation of the Coarse-Grained PDE

- Derivation of Nonlinear Diffusion

- Numerical Examples – Quantitative Agreement with Stochastic Simulations in Weaker Slowdown Regime

Stochastic Lattice Model of Pedestrian Traffic



Approach: One-dimensional $\{0, 1\}$ Lattice Configuration for

$$\sigma_k^{\pm}(t) = \begin{cases} 1, & \text{pedestrian moving to the right (left)} \\ 0, & \text{empty cell} \end{cases}$$

If no pedestrians are moving in the opposite direction: Equivalent to car traffic models

• Two pedestrians moving in the same direction cannot occupy the same cell

• Pedestrians moving into two opposite directions can occupy the same cell

To construct the microscopic cellular automata model, we consider explicit rules for the slowdown interaction.

We prescribe transition probabilities for four different pedestrian configurations in the cells neighboring to the right-moving pedestrian with $\sigma_k^+ = 1$ (assuming that $\sigma_{k+1}^+ = 0$):

$$\begin{cases} c_0 \Delta t, \text{ if } \sigma_k^- = \sigma_{k+1}^- = 0 \text{ (no left-moving pedestrians in cells } k \text{ or } k+1) \\ c_1 \Delta t, \text{ if } \sigma_k^- = 1, \sigma_{k+1}^- = 0 \text{ (a left-moving pedestrian is in cell } k) \\ c_2 \Delta t, \text{ if } \sigma_k^- = 0, \sigma_{k+1}^- = 1 \text{ (a left-moving pedestrian is in cell } k+1) \\ c_3 \Delta t, \text{ if } \sigma_k^- = \sigma_{k+1}^- = 1 \text{ (left-moving pedestrians in cells } k \text{ and } k+1) \end{cases}$$

From the common sense considerations, the velocities should obey the following relationship: $c_3 < c_2 \gtrapprox c_1 < c_0$

Transition probabilities for the left-moving pedestrian $\sigma_k^- = 1$ can be obtain in a similar manner.

Probability of a right-moving pedestrian to move from cell k to cell $k+1$ within Δt is

$$\begin{aligned} P^{+}_{k\to k+1} = \Delta t \Big[& c_0 \sigma_k^{+}(1-\sigma_{k+1}^{+})(1-\sigma_k^{-})(1-\sigma_{k+1}^{-}) \\ & + c_1 \sigma_k^{+}(1-\sigma_{k+1}^{+})\sigma_k^{-}(1-\sigma_{k+1}^{-}) \\ & + c_2 \sigma_k^{+}(1-\sigma_{k+1}^{+})(1-\sigma_k^{-})\sigma_{k+1}^{-} + c_3 \sigma_k^{+}(1-\sigma_{k+1}^{+})\sigma_k^{-}\sigma_{k+1}^{-} \Big] \end{aligned}$$

Probability of a left-moving pedestrian to move from cell k to cell $k-1$ within Δt is

$$\begin{aligned} P^{-}_{k\to k-1} = \Delta t \Big[& c_0 \sigma_k^{-}(1-\sigma_{k-1}^{-})(1-\sigma_{k-1}^{+})(1-\sigma_k^{+}) \\ & + c_1 \sigma_k^{-}(1-\sigma_{k-1}^{-})(1-\sigma_{k-1}^{+})\sigma_k^{+} \\ & + c_2 \sigma_k^{-}(1-\sigma_{k-1}^{-})\sigma_{k-1}^{+}(1-\sigma_k^{+}) + c_3 \sigma_k^{-}(1-\sigma_{k-1}^{-})\sigma_{k-1}^{+}\sigma_k^{+} \Big] \end{aligned}$$

Goal: Predict the Density of the Pedestrian Traffic, $\mathbb{E}\sigma_k^+(t)$ and $\mathbb{E}\sigma_k^-(t)$

$$
\begin{aligned}
\frac{d\mathbb{E}\sigma_k^+}{dt} = \mathbb{E}\big[& c_0\sigma_{k-1}^+(1-\sigma_k^+)(1-\sigma_{k-1}^-)(1-\sigma_k^-)) - c_0\sigma_k^+(1-\sigma_{k+1}^+)(1-\sigma_k^-)(1-\sigma_{k+1}^-) \\
& +c_1\sigma_{k-1}^+(1-\sigma_k^+)\sigma_{k-1}^-(1-\sigma_k^-) - c_1\sigma_k^+(1-\sigma_{k+1}^+)\sigma_k^-(1-\sigma_{k+1}^-) \\
& +c_2\sigma_{k-1}^+(1-\sigma_k^+)(1-\sigma_{k-1}^-)\sigma_k^- - c_2\sigma_k^+(1-\sigma_{k+1}^+)(1-\sigma_k^-)\sigma_{k+1}^- \\
& +c_3\sigma_{k-1}^+(1-\sigma_k^+)\sigma_{k-1}^-\sigma_k^- - c_3\sigma_k^+(1-\sigma_{k+1}^+)\sigma_k^-\sigma_{k+1}^-\big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\mathbb{E}\sigma_k^-}{dt} = \mathbb{E}\big[& c_0\sigma_{k+1}^-(1-\sigma_k^-)(1-\sigma_k^+)(1-\sigma_{k+1}^+) - c_0\sigma_k^-(1-\sigma_{k-1}^-)(1-\sigma_{k-1}^+)(1-\sigma_k^+) \\
& +c_1\sigma_{k+1}^-(1-\sigma_k^-)(1-\sigma_k^+)\sigma_{k+1}^+ - c_1\sigma_k^-(1-\sigma_{k-1}^-)(1-\sigma_{k-1}^+)\sigma_k^+ \\
& +c_2\sigma_{k+1}^-(1-\sigma_k^-)\sigma_k^+(1-\sigma_{k+1}^+) - c_2\sigma_k^-(1-\sigma_{k-1}^-)\sigma_{k-1}^+(1-\sigma_k^+) \\
& +c_3\sigma_{k+1}^-(1-\sigma_k^-)\sigma_k^+\sigma_{k+1}^+ - c_3\sigma_k^-(1-\sigma_{k-1}^-)\sigma_{k-1}^+\sigma_k^+\big]
\end{aligned}
$$

These equations are exact, but not closed!

Mesoscopic Model

- Notations: $\rho_k^{\pm}(t) := \mathbb{E}\sigma_k^{\pm}$
- Assumptions: $\mathbb{E}\left[\sigma_{k-1}^{+}\sigma_k^{+}\sigma_{k-1}^{-}\sigma_k^{-}\right] \approx \mathbb{E}[\sigma_{k-1}^{+}]\mathbb{E}[\sigma_k^{+}]\mathbb{E}[\sigma_{k-1}^{-}]\mathbb{E}[\sigma_k^{-}]$

$$
\begin{aligned}
\frac{d\rho_k^+}{dt} =& c_0\rho_{k-1}^+(1-\rho_k^+)(1-\rho_{k-1}^-)(1-\rho_k^-) - c_0\rho_k^+(1-\rho_{k+1}^+)(1-\rho_k^-)(1-\rho_{k+1}^-)\\
&+c_1\rho_{k-1}^+(1-\rho_k^+)\rho_{k-1}^-(1-\rho_k^-) - c_1\rho_k^+(1-\rho_{k+1}^+)\rho_k^-(1-\rho_{k+1}^-)\\
&+c_2\rho_{k-1}^+(1-\rho_k^+)(1-\rho_{k-1}^-)\rho_k^- - c_2\rho_k^+(1-\rho_{k+1}^+)(1-\rho_k^-)\rho_{k+1}^-\\
&+c_3\rho_{k-1}^+(1-\rho_k^+)\rho_{k-1}^-\rho_k^- - c_3\rho_k^+(1-\rho_{k+1}^+)\rho_k^-\rho_{k+1}^-
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\rho_k^-}{dt} =& c_0\rho_{k+1}^-(1-\rho_k^-)(1-\rho_k^+)(1-\rho_{k+1}^+) - c_0\rho_k^-(1-\rho_{k-1}^-)(1-\rho_{k-1}^+)(1-\rho_k^+)\\
&+c_1\rho_{k+1}^-(1-\rho_k^-)(1-\rho_k^+)\rho_{k+1}^+ - c_1\rho_k^-(1-\rho_{k-1}^-)(1-\rho_{k-1}^+)\rho_k^+\\
&+c_2\rho_{k+1}^-(1-\rho_k^-)\rho_k^+(1-\rho_{k+1}^+) - c_2\rho_k^-(1-\rho_{k-1}^-)\rho_{k-1}^+(1-\rho_k^+)\\
&+c_3\rho_{k+1}^-(1-\rho_k^-)\rho_k^+\rho_{k+1}^+ - c_3\rho_k^-(1-\rho_{k-1}^-)\rho_{k-1}^+\rho_k^+
\end{aligned}
$$

Macroscopic PDE Model

- $k \in \mathcal{L}$: cells with some fixed length $h > 0$ in the lattice $\mathcal{L}$
- $\Omega = [0, L]$ corresponds to $\mathcal{L}$ (the number of cells N depends on h)
- $t \to ht$ and $N \to \infty$

We rewrite the mesoscopic system in the following flux form:

$$\frac{d\rho_k^+}{dt} = -\frac{F_{k,k+1}^+ - F_{k-1,k}^+}{h}, \qquad \frac{d\rho_k^-}{dt} = \frac{F_{k,k+1}^- - F_{k-1,k}^-}{h}$$

where

$$F_{k,k+1}^+ = \rho_k^+(1-\rho_{k+1}^+)\Big[(1-\rho_{k+1}^-)\Big(c_0(1-\rho_k^-)+c_1\rho_k^-\Big) \\ +\rho_{k+1}^-\Big(c_2(1-\rho_k^-)+c_3\rho_k^-\Big)\Big]$$

$$F_{k,k+1}^- = \rho_{k+1}^-(1-\rho_k^-)\Big[(1-\rho_k^+)\Big(c_0(1-\rho_{k+1}^+)+c_1\rho_{k+1}^+\Big) \\ +\rho_k^+\Big(c_2(1-\rho_{k+1}^+)+c_3\rho_{k+1}^+\Big)\Big]$$

$$\frac{d\rho_k^+}{dt} = -\frac{F_{k,k+1}^+ - F_{k-1,k}^+}{h}, \qquad \frac{d\rho_k^-}{dt} = \frac{F_{k,k+1}^- - F_{k-1,k}^-}{h}$$

- Multiply these equations by $\varphi_k := \varphi(kh)$, where φ is a C_0^1 test function

- Use the summation by parts over Ω:

$$\sum_k \varphi_k \frac{d\rho_k^\pm}{dt} = \pm \sum_k F_{k,k+1}^\pm \frac{\varphi_{k+1} - \varphi_k}{h}$$

- Multiply by h and expand φ_{k+1} into a Taylor series about kh:

$$\sum_k \varphi_k \frac{d\rho_k^\pm}{dt} h = \pm \sum_k F_{k,k+1}^\pm [\varphi_k' + \mathcal{O}(h)] h$$

Define pedestrian densities on Ω as follows:

- Define the function $\rho^{\pm}(x,t)$ as a continuous piecewise linear interpolation (in the spatial variable) of $\rho_k^{\pm}(t)$
- Take the limit as $h \to 0+$

Due to the boundedness of both $\rho^{\pm}$ and $\frac{d\rho_k^{\pm}}{dt}$ we obtain a weak formulation of the coarse-grained model:

$$\int_{\Omega} \varphi(x) \frac{\partial}{\partial t} \rho^{\pm}(x,t)\, dx = \pm \int_{\Omega} F^{\pm}(\rho^{+}, \rho^{-}) \varphi'(x)\, dx$$

where $F^{\pm}(\rho^{+}, \rho^{-})$ are defined as the corresponding limits of $F^{\pm}_{k,k+1}$:

$$F^{+}(\rho^{+}, \rho^{-}) = f(\rho^{+}) g(\rho^{-}), \quad F^{-}(\rho^{+}, \rho^{-}) = f(\rho^{-}) g(\rho^{+})$$

where

$$f(u) = u(1-u), \quad g(u) = (c_3 - c_2 - c_1 + c_0)u^2 + (c_2 + c_1 - 2c_0)u + c_0$$

$$\int_{\Omega} \varphi(x) \frac{\partial}{\partial t} \rho^{+}(x,t)\, dx = \int_{\Omega} f(\rho^{+}) g(\rho^{-}) \varphi'(x)\, dx$$

$$\int_{\Omega} \varphi(x) \frac{\partial}{\partial t} \rho^{-}(x,t)\, dx = -\int_{\Omega} f(\rho^{-}) g(\rho^{+}) \varphi'(x)\, dx$$

Since φ is arbitrary, we have

$$\begin{cases} \rho_t^{+} + \left[f(\rho^{+}) g(\rho^{-}) \right]_x = 0 \\ \rho_t^{-} - \left[f(\rho^{-}) g(\rho^{+}) \right]_x = 0 \end{cases}$$

where

$$f(u) = u(1-u), \quad g(u) = (c_3 - c_2 - c_1 + c_0)u^2 + (c_2 + c_1 - 2c_0)u + c_0$$

Note that the velocities c_1 and c_2 enter only as a sum, and, therefore, it is not necessary to specify them separately

Properties of the PDE Model

The PDE system is only conditionally hyperbolic:

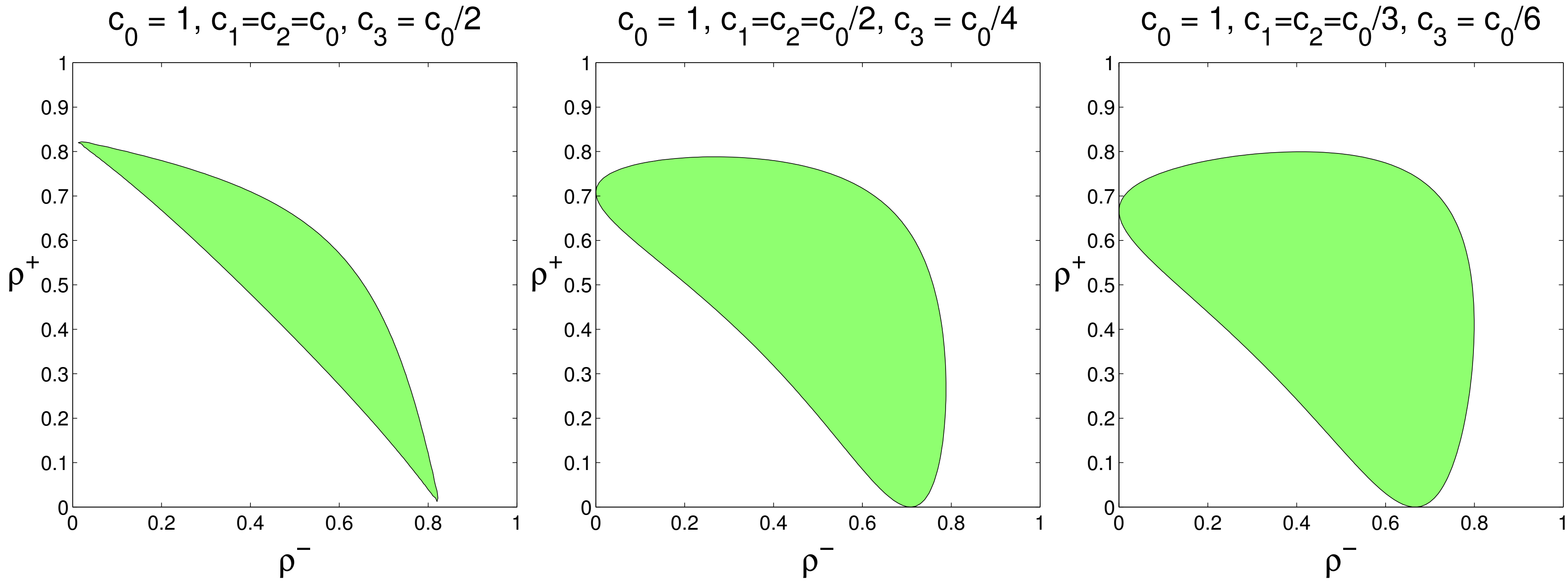
$$J(f,g) := \begin{pmatrix} f'(\rho^+)g(\rho^-) & f(\rho^+)g'(\rho^-) \\ -f(\rho^-)g'(\rho^+) & -f'(\rho^-)g(\rho^+) \end{pmatrix}$$

has real eigenvalues only if

$$\left[f'(\rho^-)g(\rho^+) + f'(\rho^+)g(\rho^-)\right]^2 - 4f(\rho^-)f(\rho^+)g'(\rho^-)g'(\rho^+) > 0$$

- For any particular choice of the velocities c_0, c_1, c_2 and c_3 there is a region on non-hyperbolicity in the (ρ^-, ρ^+) plane

- The non-hyperbolicity can only manifest itself when pedestrians moving in the opposite directions are both present in a particular location



- The non-hyperbolic region depends only on the ratio of velocities c_1/c_0, c_2/c_0 and c_3/c_0, but not on the particular value of c_0
- The non-hyperbolic region becomes larger as the slowdown effect becomes more pronounces (i.e., as the ratios c_1/c_0, c_2/c_0 and c_3/c_0 become smaller)
- The loss of hyperbolicity may induce instabilities, which are nonphysical and can be removed by introducing a nonlinear diffusive correction to the system

Nonlinear Diffusive Correction

$$\frac{d\rho_k^+}{dt} = -\frac{F_{k,k+1}^+ - F_{k-1,k}^+}{h}, \qquad \frac{d\rho_k^-}{dt} = \frac{F_{k,k+1}^- - F_{k-1,k}^-}{h}$$

where

$$F_{k,k+1}^+ = \rho_k^+(1-\rho_{k+1}^+)\Big[(1-\rho_{k+1}^-)\Big(c_0(1-\rho_k^-) + c_1\rho_k^-\Big) \\ +\rho_{k+1}^-\Big(c_2(1-\rho_k^-) + c_3\rho_k^-\Big)\Big]$$

$$F_{k,k+1}^- = \rho_{k+1}^-(1-\rho_k^-)\Big[(1-\rho_k^+)\Big(c_0(1-\rho_{k+1}^+) + c_1\rho_{k+1}^+\Big) \\ +\rho_k^+\Big(c_2(1-\rho_{k+1}^+) + c_3\rho_{k+1}^+\Big)\Big]$$

The derivation of the coarse-grained PDE system can also be obtained by formally using the Taylor expansions

$$\rho_{k\pm 1}^{\pm} = \rho_k^{\pm} \pm h(\rho_k^{\pm})' + \frac{h^2}{2}(\rho_k^{\pm})'' + \mathcal{O}(h^3)$$

followed by passing to the limit as $h \to 0+$

- Keep h fixed
- Neglect the $\mathcal{O}(h^3)$ terms

$$\begin{aligned}\rho_t^+ + \Big[f(\rho^+)g(\rho^-)\Big]_x = h\Big[&\frac{c_0}{2}\rho_{xx}^+ + (c_1 - c_0 + (c_3 - c_2 - c_1 + c_0)\rho^- \\ &+ (c_2 - c_1)\rho^+)\rho_x^-\rho_x^+ + \frac{1}{2}(c_1 - c_2)\rho^+(1-\rho^+)\rho_{xx}^- \\ &+ \frac{1}{2}\left((c_1 + c_2 - 2c_0)\rho^- + (c_3 - c_2 - c_1 + c_0)(\rho^-)^2\right)\rho_{xx}^+\Big]\end{aligned}$$

$$\begin{aligned}\rho_t^- - \Big[f(\rho^-)g(\rho^+)\Big]_x = h\Big[&\frac{c_0}{2}\rho_{xx}^- + (c_1 - c_0 + (c_3 - c_2 - c_1 + c_0)\rho^+ \\ &+ (c_2 - c_1)\rho^-)\rho_x^+\rho_x^- + \frac{1}{2}(c_1 - c_2)\rho^-(1-\rho^-)\rho_{xx}^+ \\ &+ \frac{1}{2}\left((c_2 + c_1 - 2c_0)\rho^+ + (c_3 - c_2 - c_1 + c_0)(\rho^+)^2\right)\rho_{xx}^-\Big]\end{aligned}$$

- Replace h with a small parameter ε and use the formulae for the fluxes

$$f(u) = u(1-u), \quad g(u) = (c_3 - c_2 - c_1 + c_0)u^2 + (c_2 + c_1 - 2c_0)u + c_0$$

to obtain

$$\begin{cases} \rho_t^+ + \left[f(\rho^+)g(\rho^-)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^-)\rho_x^+ + (c_1 - c_2)f(\rho^+)\rho_x^-\right]_x \\ \rho_t^- - \left[f(\rho^-)g(\rho^+)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^+)\rho_x^- + (c_1 - c_2)f(\rho^-)\rho_x^+\right]_x \end{cases}$$

- The coefficients of the nonlinear diffusion are positive provided $c_1 \geq c_2$ and both ρ^+ and ρ^- are between 0 and 1

- Further simplifying assumption $c_1 = c_2$ leads to

$$\begin{cases} \rho_t^+ + \left[f(\rho^+)g(\rho^-)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^-)\rho_x^+\right]_x \\ \rho_t^- - \left[f(\rho^-)g(\rho^+)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^+)\rho_x^-\right]_x \end{cases}$$

- The assumption $c_1 = c_2$ is rather mild since the velocities c_1 and c_2 only enter as a sum into the fluxes

$$\begin{cases} \rho_t^+ + \left[f(\rho^+)g(\rho^-)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^-)\rho_x^+\right]_x \\ \rho_t^- - \left[f(\rho^-)g(\rho^+)\right]_x = \frac{\varepsilon}{2}\left[g(\rho^+)\rho_x^-\right]_x \end{cases}$$

- The nonlinear diffusion reflects the presence of pedestrians moving in the opposite direction

– If $\rho^- = 0$ (i.e., no pedestrians moving to the left are present), the diffusion reduces to the usual linear diffusion $0.5\varepsilon c_0 \rho_{xx}^+$

– If $\rho^- = 1$, then the diffusion also becomes linear $0.5\varepsilon c_3 \rho_{xx}^+$, but with a smaller coefficient (since $c_3 < c_0$) reflecting a high density presence of the pedestrians moving in the opposite direction

- ε needs to be established experimentally

Semi-Discrete Central-Upwind Scheme

$$\boldsymbol{\rho}_t + \mathbf{F}(\boldsymbol{\rho})_x = (Q(\boldsymbol{\rho})\boldsymbol{\rho}_x)_x$$

- $\boldsymbol{\rho} := (\rho^+, \rho^-)^T$
- $\mathbf{F}(\boldsymbol{\rho}) := \Big(f(\rho^+)g(\rho^-), f(\rho^-)g(\rho^+)\Big)^T$
- $Q(\boldsymbol{\rho}) = \frac{\varepsilon}{2}\begin{pmatrix} g(\rho^-) & (c_1 - c_2)f(\rho^+) \\ (c_1 - c_2)f(\rho^-) & g(\rho^+) \end{pmatrix}$
- $\overline{\boldsymbol{\rho}}_j^n \approx \frac{1}{\Delta x} \int\limits_{C_j} \boldsymbol{\rho}(x, t^n)\, dx$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$
- The solution is approximated by a piecewise linear conservative, second-order accurate, non-oscillatory reconstruction:

$$\widetilde{\boldsymbol{\rho}}^n(x) = \overline{\boldsymbol{\rho}}_j^n + (\boldsymbol{\rho}_x)_j^n(x - x_j) \quad \text{for } x \in C_j$$

The solution is evolved by the semi-discrete central-upwind scheme

[Kurganov, Tadmor; 2000]

[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Lin; 2007]

$$\frac{d\bar{\boldsymbol{\rho}}_j(t)}{dt} = -\frac{\mathbf{H}_{\mathbf{j}-\frac{1}{2}}(\mathbf{t}) - \mathbf{H}_{\mathbf{j}+\frac{1}{2}}(\mathbf{t})}{\Delta x} + \frac{\mathbf{P}_{\mathbf{j}-\frac{1}{2}}(\mathbf{t}) - \mathbf{P}_{\mathbf{j}+\frac{1}{2}}(\mathbf{t})}{\Delta x}$$

with the numerical fluxes given by

$$\mathbf{H}_{\mathbf{j}+\frac{1}{2}} = \frac{a^{+}_{j+\frac{1}{2}}\mathbf{F}(\boldsymbol{\rho}^{\mathsf{E}}_j) - a^{-}_{j+\frac{1}{2}}\mathbf{F}(\boldsymbol{\rho}^{\mathsf{W}}_{j+1})}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}} + \frac{a^{+}_{j+\frac{1}{2}}a^{-}_{j+\frac{1}{2}}}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}}\left[\boldsymbol{\rho}^{\mathsf{W}}_{j+1} - \boldsymbol{\rho}^{\mathsf{E}}_j\right]$$

$$\mathbf{P}_{\mathbf{j}+\frac{1}{2}} = Q(\boldsymbol{\rho}_{j+\frac{1}{2}})\frac{\bar{\boldsymbol{\rho}}_{j+1} - \bar{\boldsymbol{\rho}}_j}{\Delta x}, \quad \boldsymbol{\rho}_{j+\frac{1}{2}} = \frac{\boldsymbol{\rho}^{\mathsf{E}}_j + \boldsymbol{\rho}^{\mathsf{W}}_{j+1}}{2}$$

The reconstructed point values are

$$\boldsymbol{\rho}_j^{\mathsf{E}} := \bar{\boldsymbol{\rho}}_j + \frac{\Delta x}{2}(\boldsymbol{\rho}_x)_j, \quad \boldsymbol{\rho}_j^{\mathsf{W}} := \bar{\boldsymbol{\rho}}_j - \frac{\Delta x}{2}(\boldsymbol{\rho}_x)_j$$

The discontinuities appearing at the reconstruction step at the interface points $\{x_{j+\frac{1}{2}}\}$ propagate at finite speeds estimated by

$$a^{+}_{j+\frac{1}{2}} := \max\left\{\lambda_2\Big(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\boldsymbol{\rho}^{\mathsf{E}}_{j+\frac{1}{2}})\Big), \lambda_2\Big(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\boldsymbol{\rho}^{\mathsf{W}}_{j+\frac{1}{2}})\Big), 0\right\}$$

$$a^{-}_{j+\frac{1}{2}} := \min\left\{\lambda_1\Big(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\boldsymbol{\rho}^{\mathsf{E}}_{j+\frac{1}{2}})\Big), \lambda_1\Big(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\boldsymbol{\rho}^{\mathsf{W}}_{j+\frac{1}{2}})\Big), 0\right\}$$

$\lambda_1 < \lambda_2$: eigenvalues of the Jacobian $\frac{\partial \mathbf{F}}{\partial \boldsymbol{\rho}}$

The eigenvalues of the Jacobian matrix are calculated as follows...

We denote by

$$R = f'(\rho^+)g(\rho^-) - f'(\rho^-)g(\rho^+)$$
$$D = \left[f'(\rho^-)g(\rho^+) + f'(\rho^+)g(\rho^-)\right]^2 - 4f(\rho^-)f(\rho^+)g'(\rho^-)g'(\rho^+)$$

and consider the two possible cases:

- If both $D_j^{\mathrm{E}} \geq 0$ and $D_{j+1}^{\mathrm{W}} \geq 0$ (hyperbolic regime), then

$$a^+_{j+\frac{1}{2}} = \frac{1}{2}\max\left\{R_j^{\mathrm{E}} + \sqrt{D_j^{\mathrm{E}}},\, R_{j+1}^{\mathrm{W}} + \sqrt{D_{j+1}^{\mathrm{W}}},\, 0\right\}$$
$$a^-_{j+\frac{1}{2}} = \frac{1}{2}\min\left\{R_j^{\mathrm{E}} - \sqrt{D_j^{\mathrm{E}}},\, R_{j+1}^{\mathrm{W}} - \sqrt{D_{j+1}^{\mathrm{W}}},\, 0\right\}$$

- If either $D_j^{\mathrm{E}} < 0$ or $D_{j+1}^{\mathrm{W}} < 0$ (nonhyperbolic regime), then

$$a^+_{j+\frac{1}{2}} = \frac{1}{2}\max\left\{\sqrt{(R_j^{\mathrm{E}})^2 - D_j^{\mathrm{E}}},\, \sqrt{(R_{j+1}^{\mathrm{W}})^2 - D_{j+1}^{\mathrm{W}}}\right\}$$
$$a^-_{j+\frac{1}{2}} = -a^+_{j+\frac{1}{2}}$$

- The choice of one-sided local speeds in the nonhyperbolic regime is ad-hoc

- We have not tried to stabilize the inviscid PDE solution by increasing the amount of numerical viscosity

- The solution has been stabilized by adding nonlinear diffusion terms rigorously derived from the mesoscopic formulation

Example — "Red Light" Initial Conditions

Two (relatively small) groups of pedestrians standing still and starting to move toward each other at time $t = 0$:

- Microscopic model:

$$c_0 = 0.8m/s,\ c_1 = c_2 = c_0/a,\ c_3 = c_0/(2a), \quad a = 2 \quad \text{or} \quad a = 3$$

$$\sigma^+(k,0) = \begin{cases} 1, & n_1 \le k \le n_2 \\ 0, & \text{otherwise} \end{cases} \qquad \sigma^-(k,0) = \begin{cases} 1, & N - n_2 \le k \le N - n_1 \\ 0, & \text{otherwise} \end{cases}$$

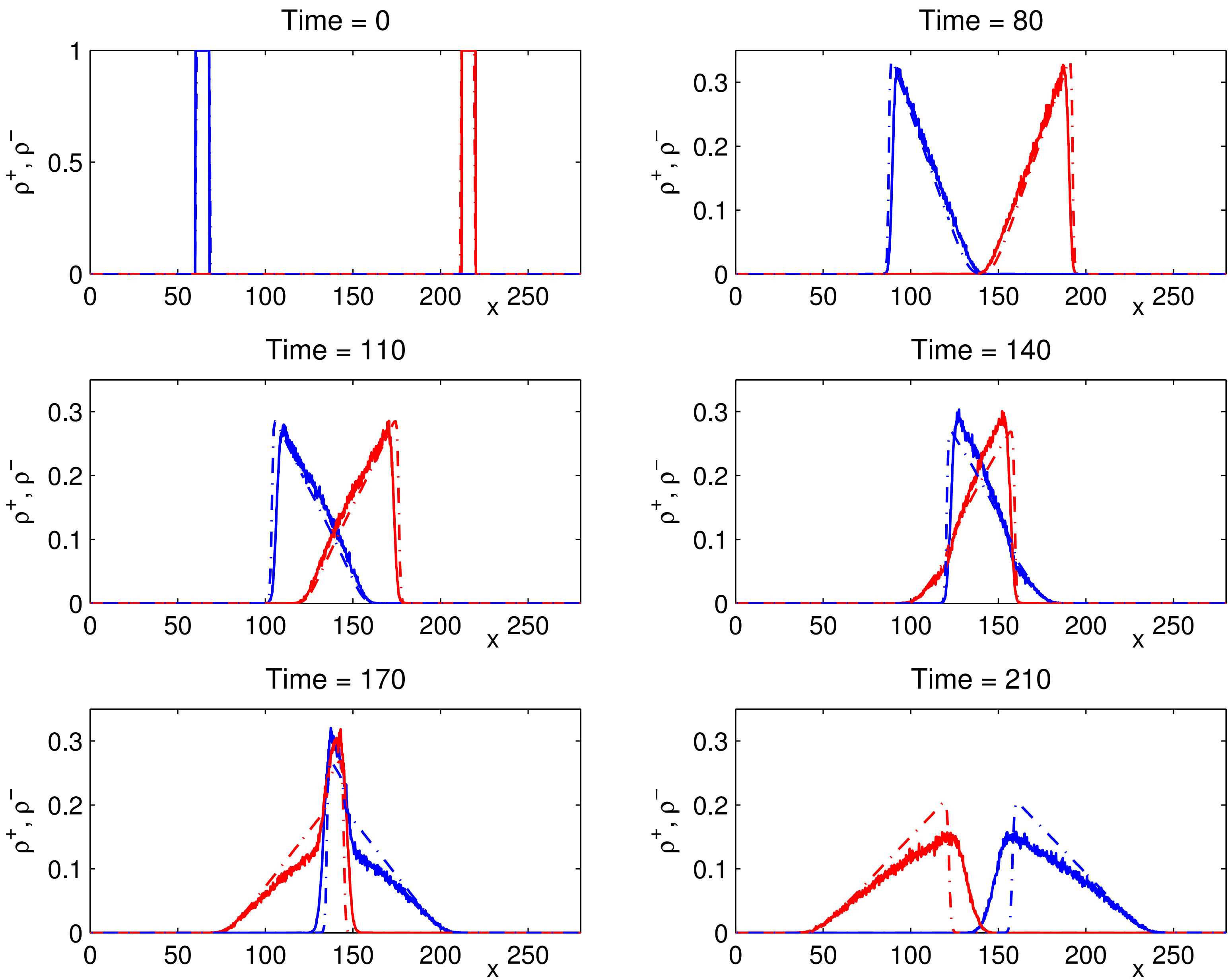
$$N = 1400,\ n_1 = 301,\ n_2 = 340,\ h = 0.2m,\ \Delta t = 0.01s,\ MC = 5000$$

- Macroscopic model:

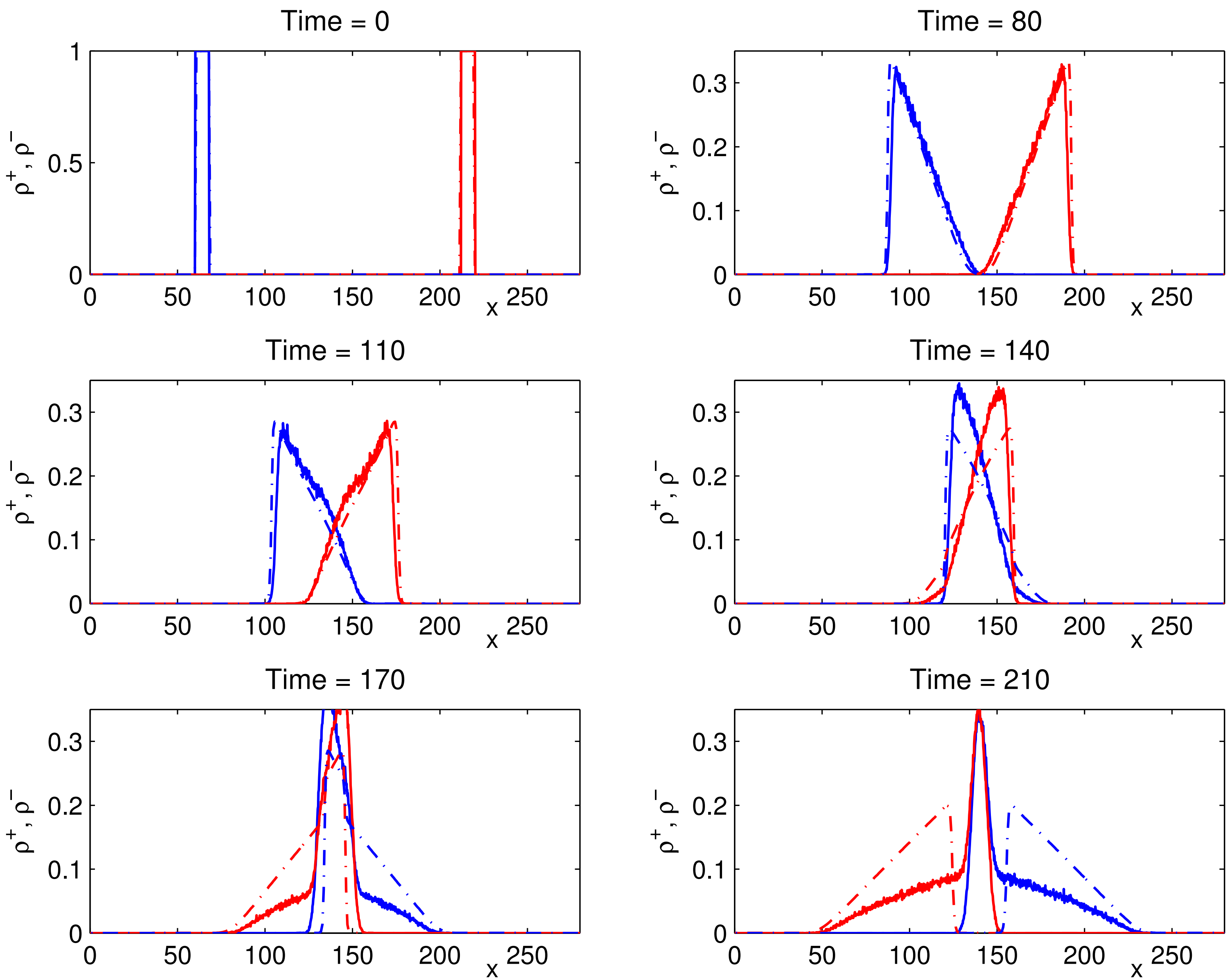
$$\rho^+(x,0) = \begin{cases} 1, & 60 < x < 68 \\ 0, & \text{otherwise} \end{cases} \qquad \rho^-(x,0) = \begin{cases} 1, & 212 < x < 220 \\ 0, & \text{otherwise} \end{cases}$$

$$L = 280,\ \Delta x = 0.8$$

$a = 2$



$a = 3$


Time = 0
Time = 80
Time = 110
Time = 140
Time = 170
Time = 210
ρ^+, ρ^-
x
0
50
100
150
200
250
0.5
1
0.1
0.2
0.3

Example — Fully Mixed Initial Conditions

Pedestrian movement in a periodic domain, which is divided into 30 sectors with 15 cells in each sector (totally $N = 450$ cells)

- Microscopic model:

– the total number of 70 pedestrians (35 moving in each direction)

– the initial distribution of pedestrians is purely random (uniform)

– sectors are $7m$ long; $L = 210m$; $\Delta t = 0.005$, $MC = 3000$

- Macroscopic model:

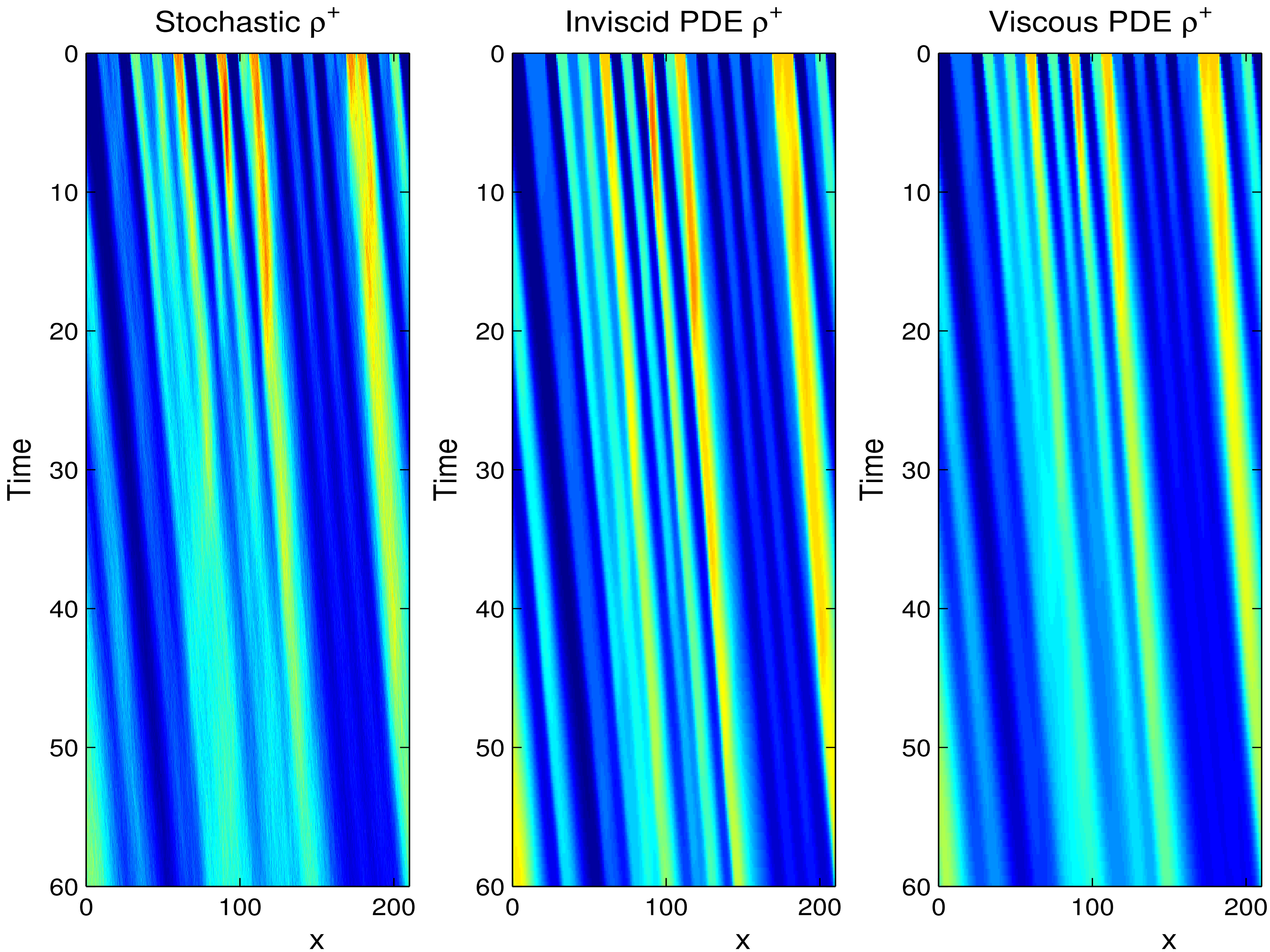
$$\rho^{\pm}(x,0) = \frac{n_i^{\pm}}{15} \quad \text{for} \quad \frac{i-1}{30}L < x < \frac{i}{30}L, \quad i = 1, \dots, 30$$

– n_i^+ and n_i^-: right- and left-moving pedestrians in the ith sector

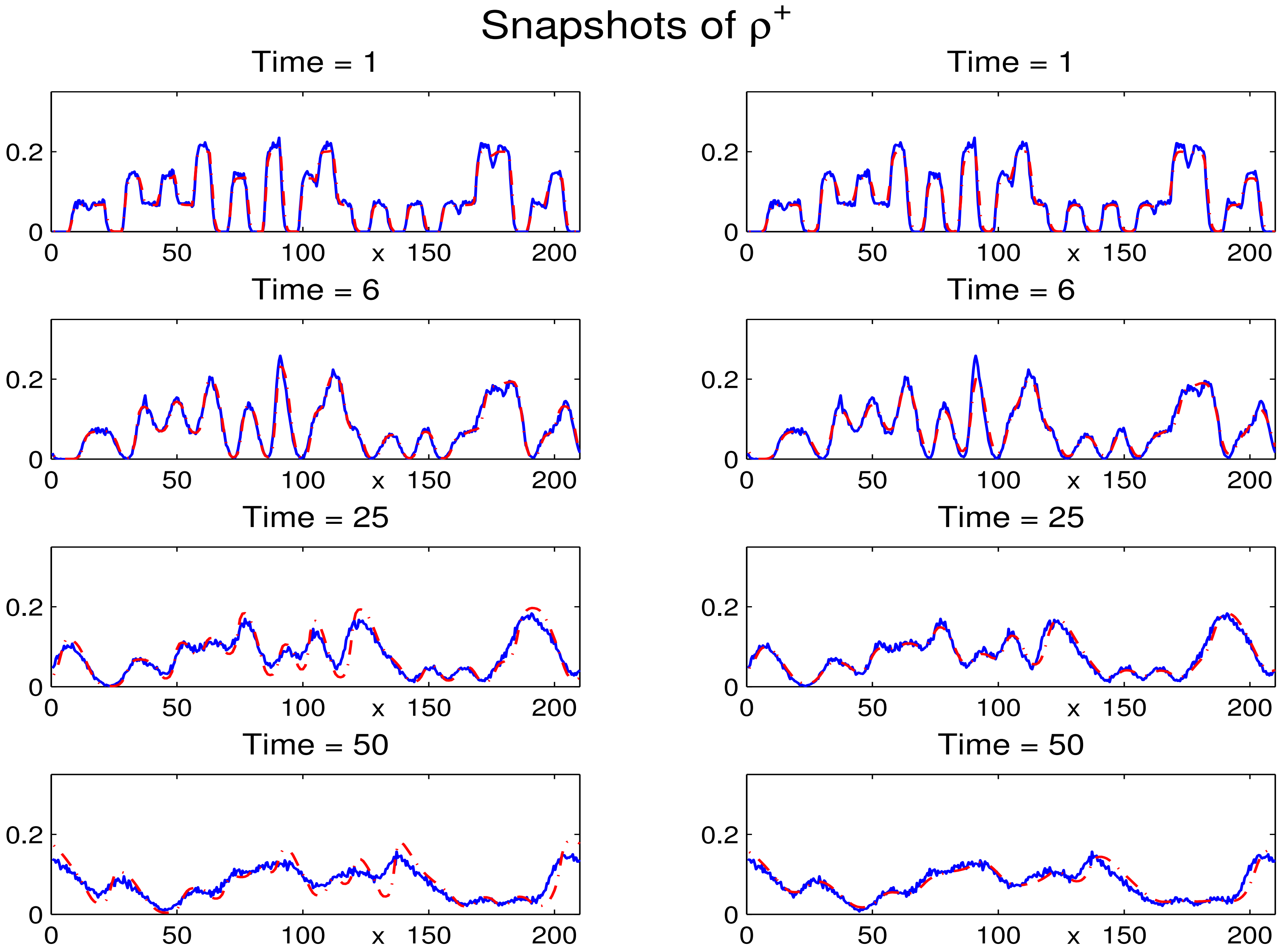
– $c_0 = 1m/s, \quad c_1 = c_2 = c_0/a, \quad c_3 = c_0/(2a), \quad a = 2 \quad \text{or} \quad a = 3$

– $L = 210, \quad \Delta x = 1, \quad \varepsilon = 0.5$

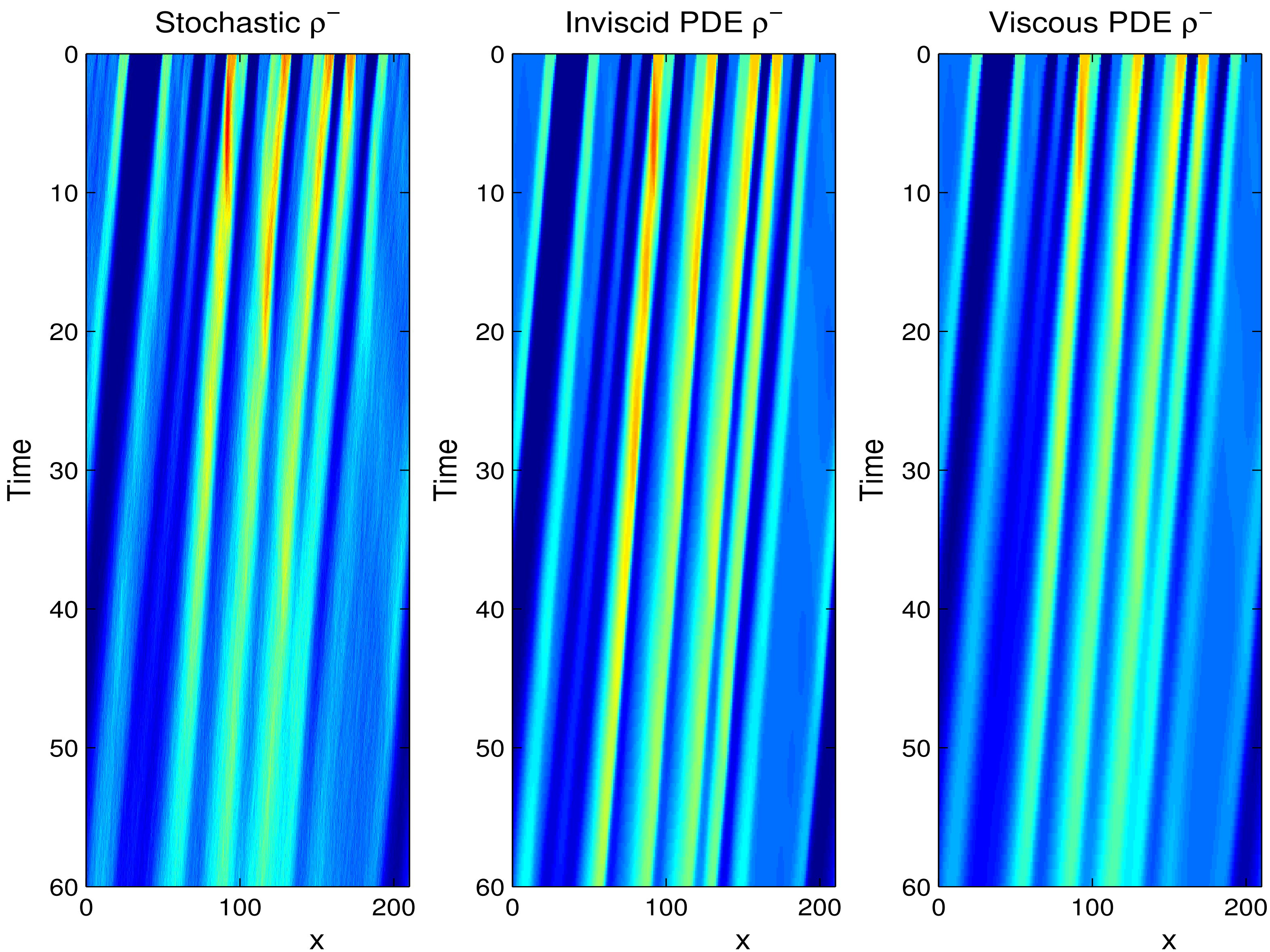
$a = 2$ (right-moving pedestrians)



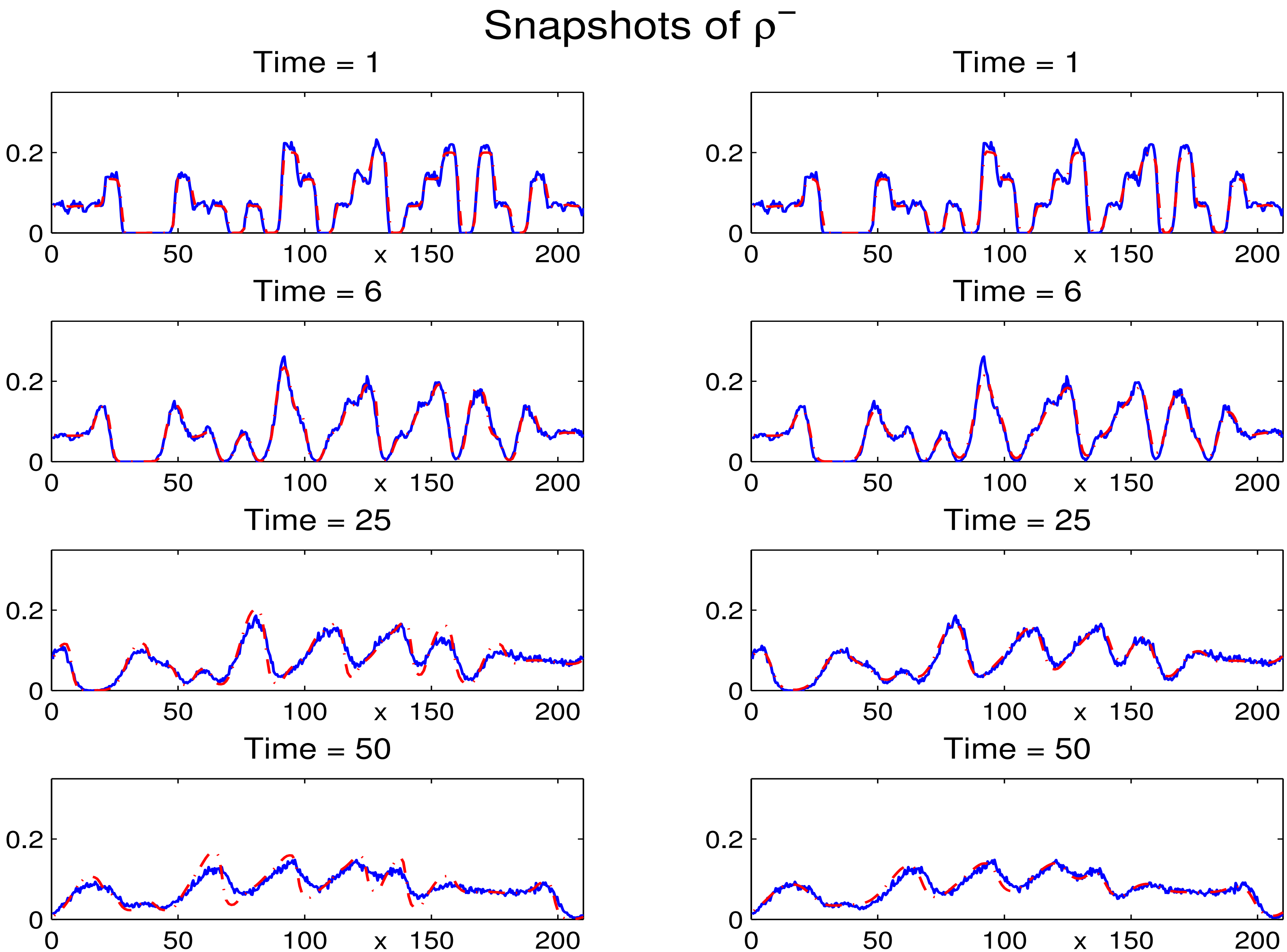
$a = 2$ (right-moving pedestrians)



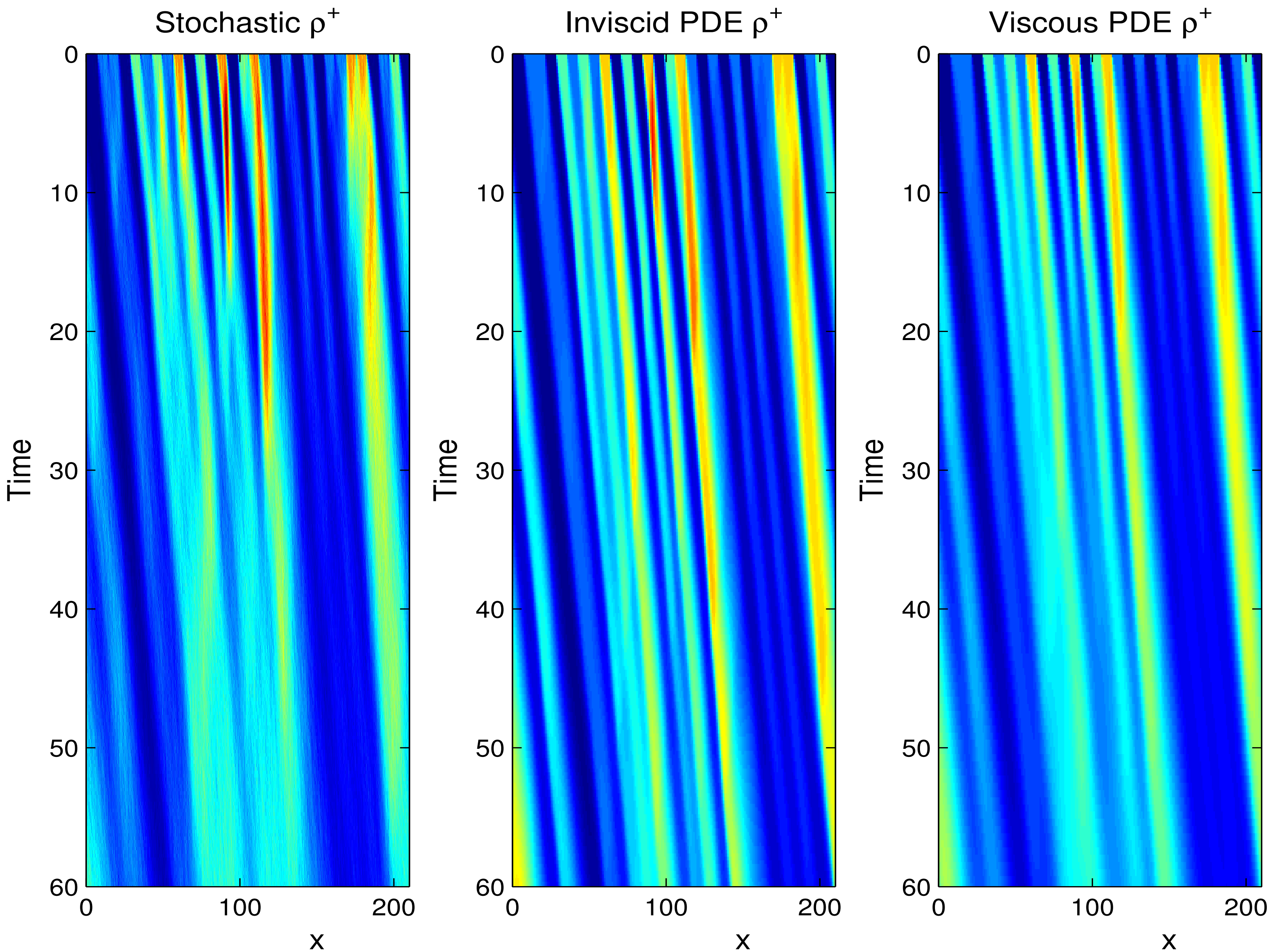
$a = 2$ (left-moving pedestrians)



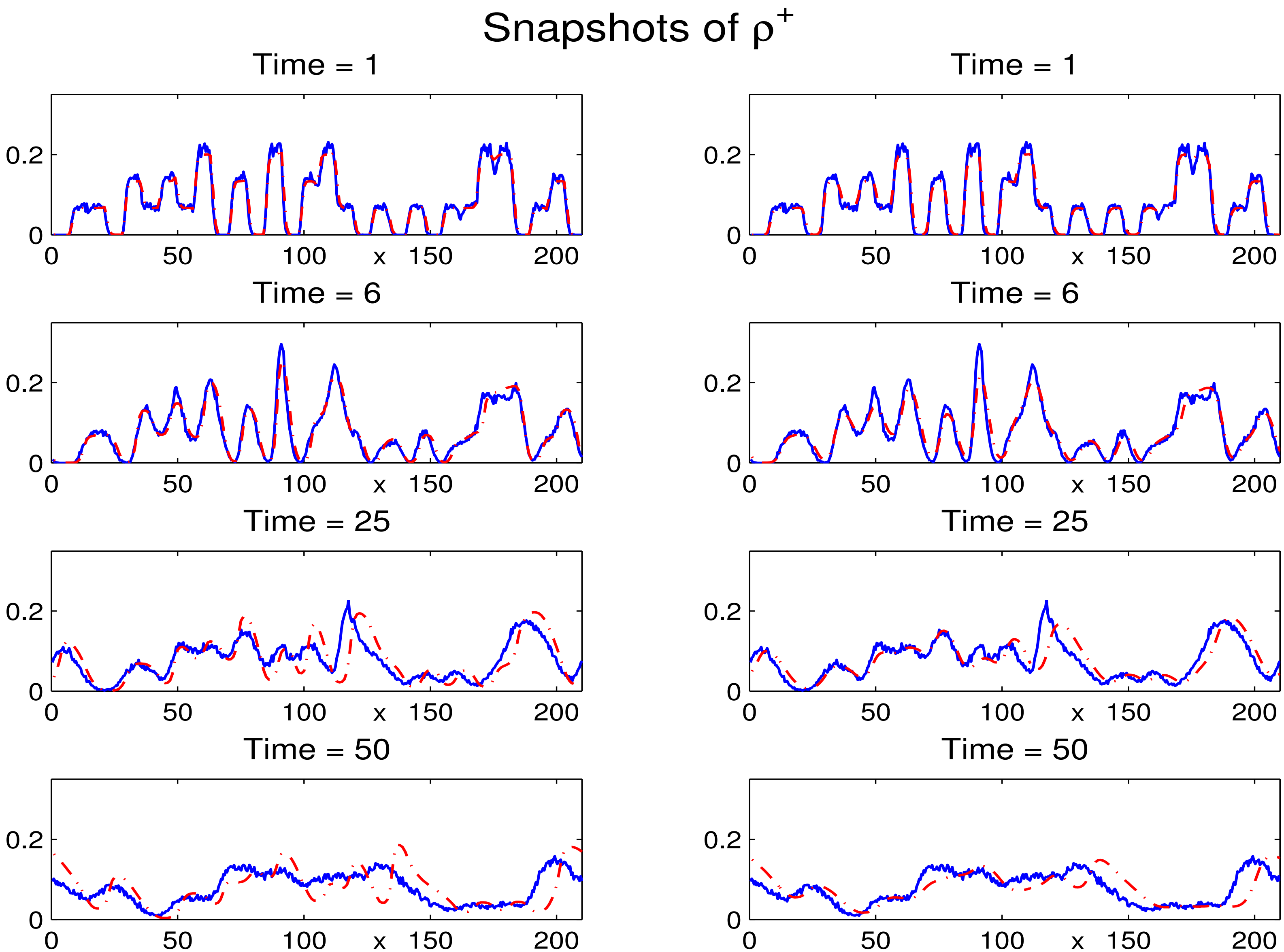
$a = 2$ (left-moving pedestrians)



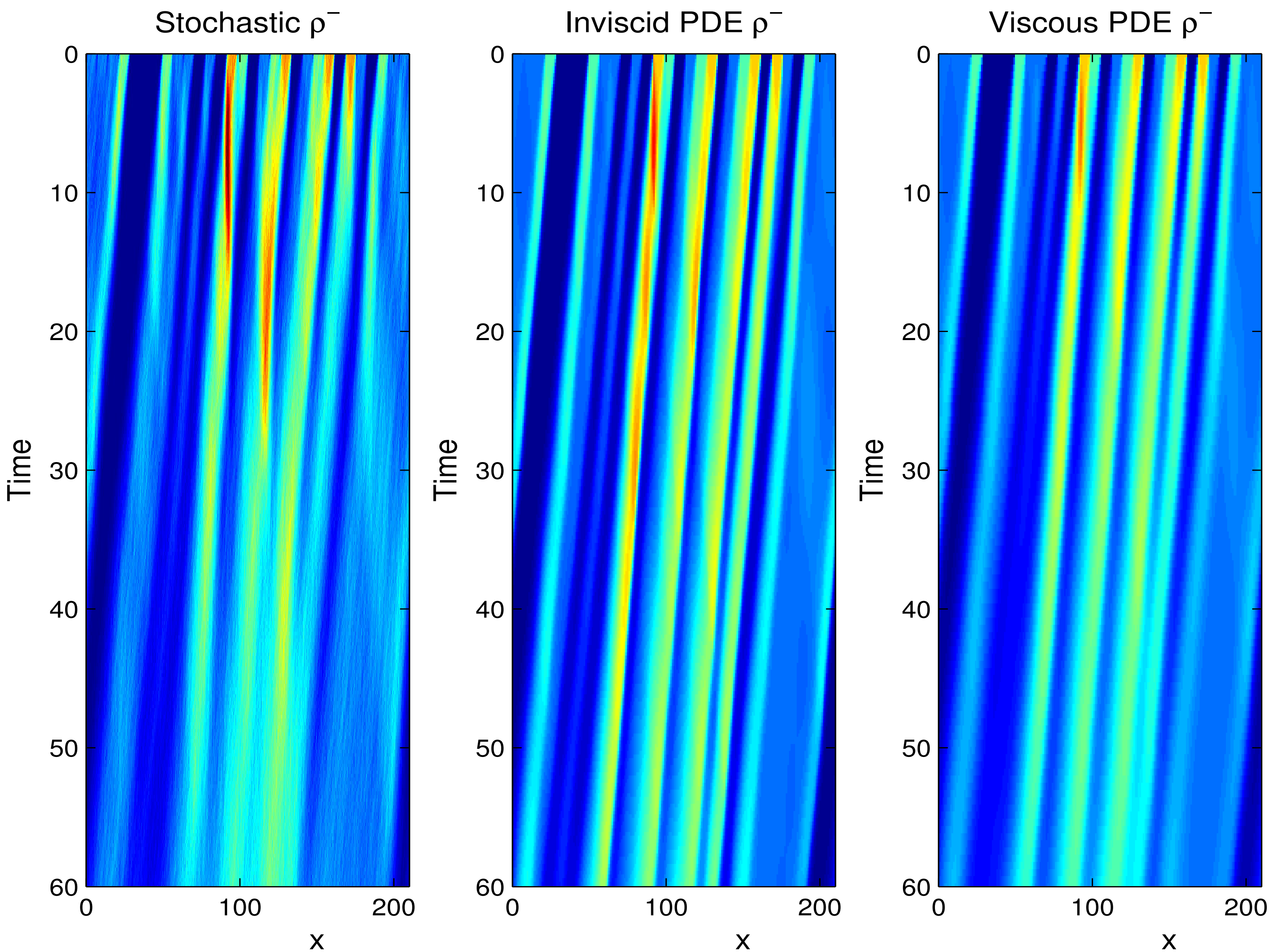
$a = 3$ (right-moving pedestrians)



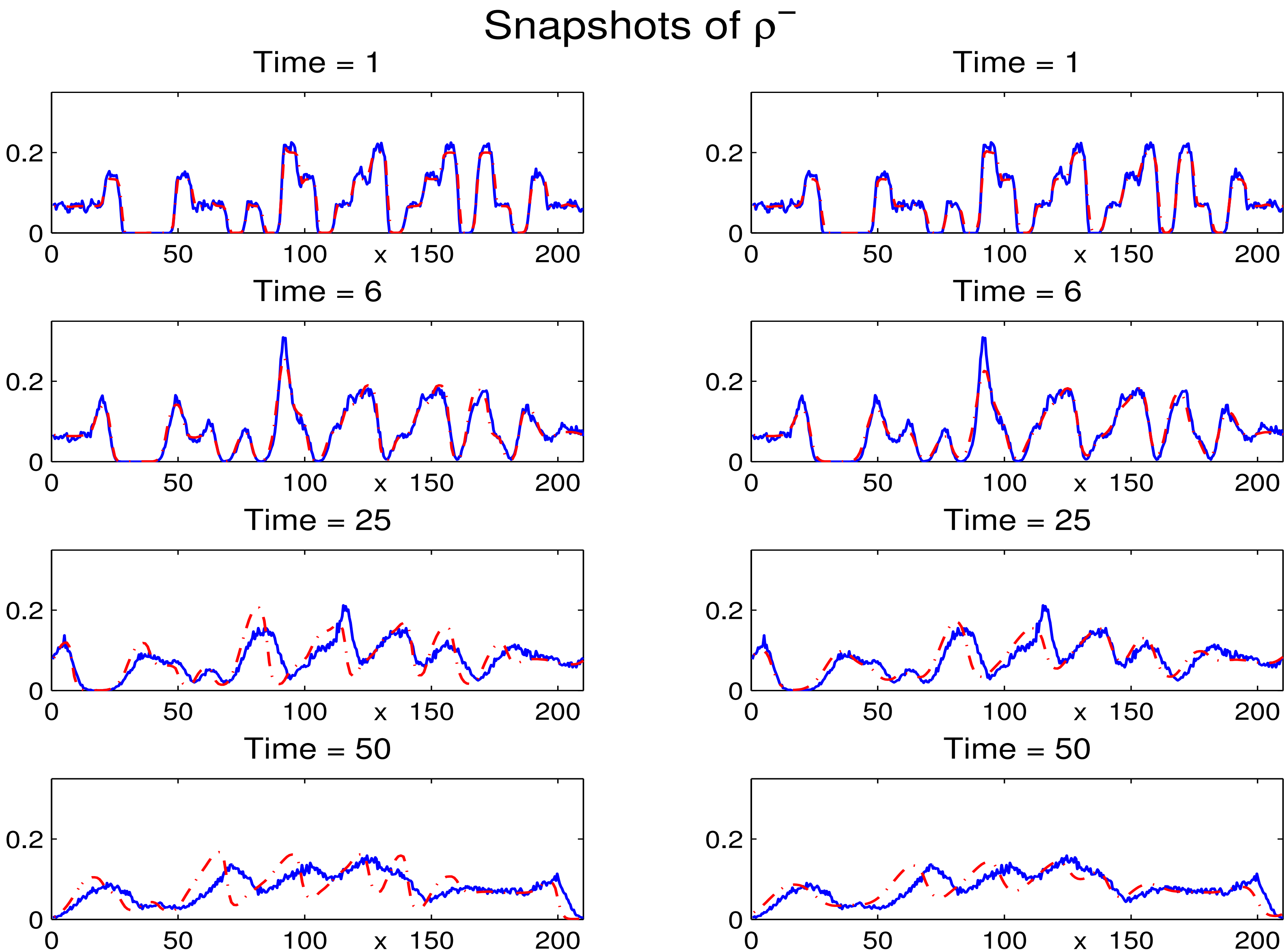
$a = 3$ (right-moving pedestrians)



$a = 3$ (left-moving pedestrians)



$a = 3$ (left-moving pedestrians)



Example — Nonhyperbolic Regime

Pedestrian movement in a periodic domain with velocities

$$c_0 = 1m/s, \quad c_1 = c_2 = c_0/a, \quad c_3 = c_0/(2a), \quad a = 2 \quad \text{or} \quad a = 3$$

- Microscopic model:

– the number of cells is $N = 900$

– the cell size is $420/900 \approx 0.4667m$

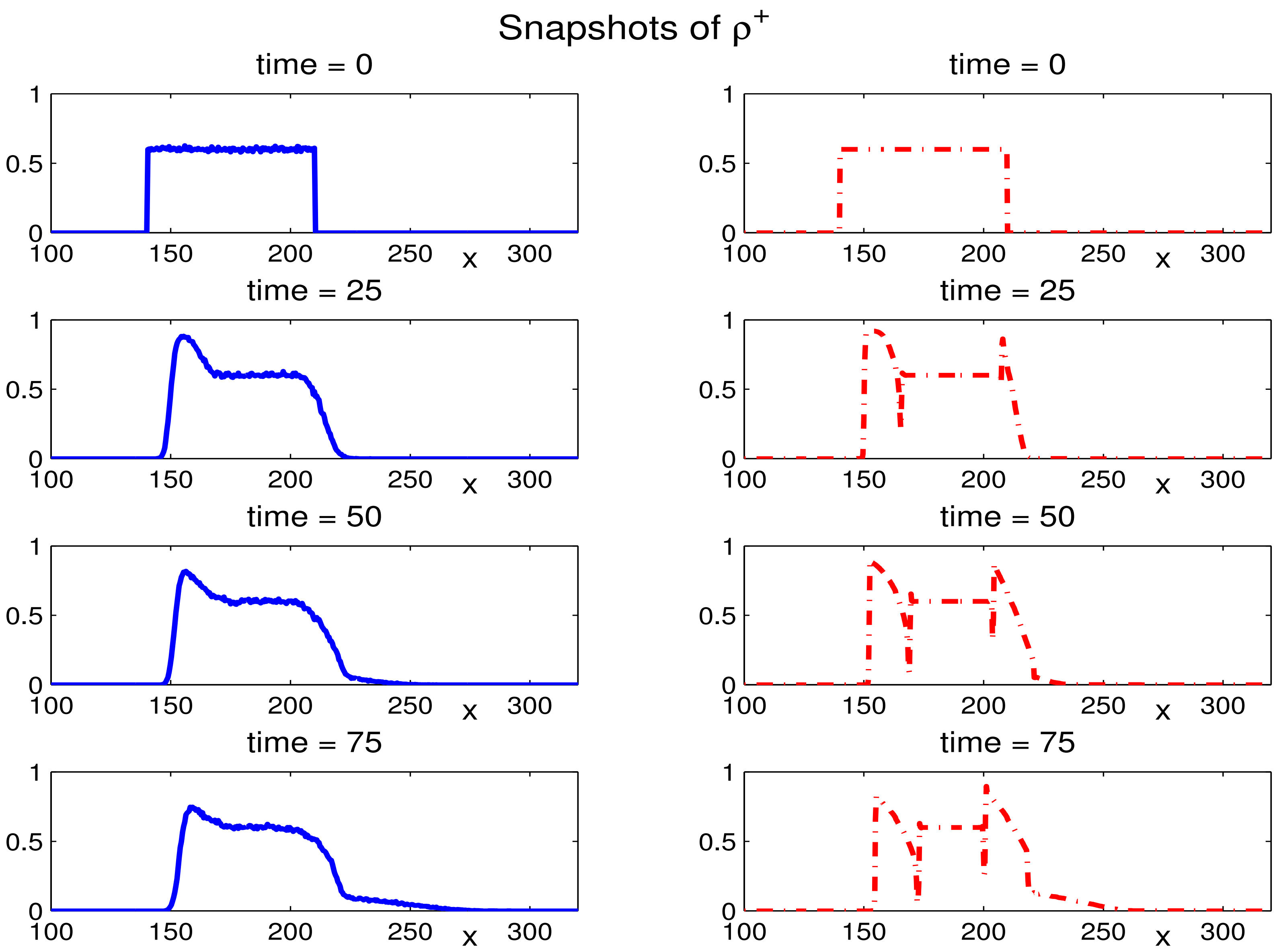
– the time step is $\Delta t = 0.005$ and $MC = 3000$

- Macroscopic model:

$$\rho^+(x,0) = \begin{cases} 0.6, & 140 \le x \le 210 \\ 0, & \text{otherwise} \end{cases} \quad \rho^-(x,0) = \begin{cases} 0.6, & 186.6 \le x \le 233.3 \\ 0, & \text{otherwise} \end{cases}$$

$$L = [0, 420], \qquad \Delta x = 420/1280$$

$a = 2$ (right-moving pedestrians), CA vs. inviscid PDE



$a = 2$ (right-moving pedestrians), CA vs. viscous PDE, $\varepsilon = 0.5, 1.5$