The N-player War of Attrition

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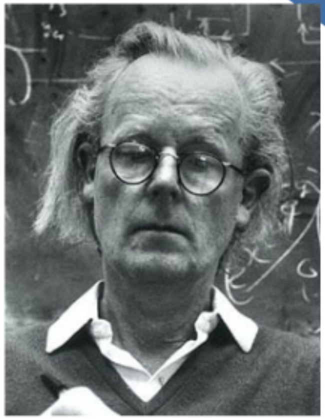
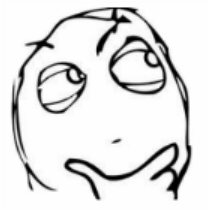
Introduction

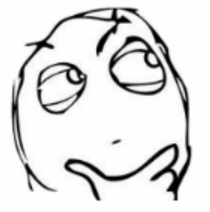
John Maynard Smith (1974)



2 Players:

Time cost = -t





Prize = V > 0

Payoff-function

$$\mathcal{J}_x(\tau_x, \tau_y) := \begin{cases} V - \tau_y, & \text{if } \tau_x > \tau_y \\ -\tau_x, & \text{if } \tau_x < \tau_y. \end{cases}$$

How to play?

Pure strategy

Mixed strategy

$t \in \mathbb{R}_+$

$\mu(dt) \in \mathcal{M}_1(\mathbb{R}_+)$

The 2-player case

In 1976 Bishop and Cannings showed that the classical 2-player War of Attrition admits a unique ESS, namely:

$$\mu(dt) = \frac{1}{V} e^{-x/V}$$

(having a very long tail!?)

Recall: the mixed strategy μ is an ESS (Evolutionary Stable Strategy) if and only if either

$$* \quad \mathcal{J}(\mu, \mu) > \mathcal{J}(\mu, \pi)$$

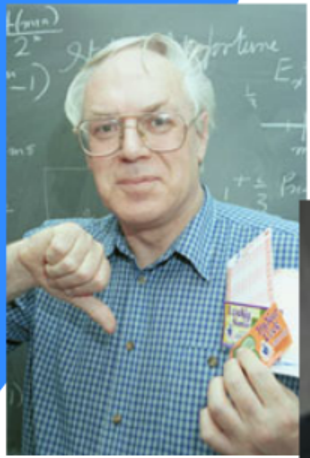
for any other mixed strategy π or, if "=" for some $\hat{\pi}$, then

$$* \quad \mathcal{J}(\mu, \hat{\pi}) > \mathcal{J}(\hat{\pi}, \hat{\pi})$$

Prezi

N-player generalizations

John Haigh



Chriss Cannings

"The n-person War of Attrition" (1989)

→ Dynamic Model

→ Static Model

The Dynamic N-player model

N available prizes: $\{V_k\}_{k=1}^{N}$

1st round

(i) Each of the N players choose a waiting time.

(ii) The player having the least time receives the prize V_1, pays the waiting time cost and leaves the game.

(iii) The remaining players pay the same time cost and proceeds to the next round.

2nd round → … … → (N-1)'th round

The Static N-player model

N available prizes: $\{V_k\}_{k=1}^{N}$

One-Shot game

(i) Each of the N players choose a waiting time.

(ii) The prizes are handed out according to the order of the chosen waiting times, i.e. the player with the least waiting time receives V_1 and so forth.

(iii) All players pay their individual waiting time.

How to play in these models?

Evolutionary Stable Strategy (ESS)

A mixed strategy μ^* is an N-player ESS if either

(i) $\mathcal{J}_N(\mu^*|\mu^*, ..., \mu^*) > \mathcal{J}_N(\mu|\mu^*, ..., \mu^*)$

or, if "=" in (i) for some $\bar{\mu}$, then

(ii) $\mathcal{J}_N(\mu^*|\mu^*, ..., \bar{\mu}) > \mathcal{J}_N(\bar{\mu}|\mu^*, ..., \bar{\mu})$

Note: An ESS is also a Nash-equilibrium, but the opposite is false!

ESS in the N-player War of Attrition?

The dynamic model always has a unique ESS!

Play: $$\mu(d\tau) = \frac{1}{(N-k)(V_{k+1}-V_k)} \exp\left\{-\frac{\tau}{(N-k)(V_{k+1}-V_k)}\right\} d\tau$$

in round (k + 1).

The static model ... ... has a ... ... ESS?

Ex:

(i) If $\{V_k\}_{k=1}^{N}$ linj. increasing there is a unique ESS.

(ii) If $V_1 = 1, V_2 = 4, V_3 = 6$ there is a candidate ESS, but it is not! (it is a Nash-equilibrium)

(iii) If $V_1 = 1, V_2 = 2, V_3 = 1$ there is not even a Nash-equilibrium.

Consider the limit when N tends to infinity!

The Dynamic Model:

The "game evolution" can be seen as a C.T.M.C

$$X(t) = \sum_{k=1}^{N-1} \frac{1}{N} \mathbb{I}_{\{T_1+\ldots+T_k \leq t\}}, \quad T_k \sim \exp\left(\frac{N-k+1}{(N-k)\,(V_{k+1}-V_k)}\right)$$

$X(t)$: 0 1/N (N-2)/N (N-1)/N

and after some calculations one finds that

$$\Rightarrow \mathbb{E}[X(t)] = \sum_{i=1}^{N-1} \frac{i}{N} \sum_{l=1}^{i+1} \frac{\prod_{k=1}^{i} \lambda_k}{\prod_{k=1,k\neq l}^{i+1} (\lambda_k - \lambda_l)} \cdot e^{-\lambda_l t}$$

A useful lemma: Let $\{\lambda_i\}_{i=1}^n$ be a sequence of positive and distinct real numbers. Then, if $f_i(t) = \lambda_i e^{-\lambda_i t}\chi_{[0,\infty)}$, it holds that

$$f_1 * f_2 * \ldots * f_n(t) = \sum_{l=1}^{n} \frac{\prod_{k=1}^n \lambda_k}{\prod_{k=1,k\neq l}^n (\lambda_k - \lambda_l)} \cdot e^{-\lambda_l t}$$

Consider $\mathcal{L}\left(\mathbb{E}[X(t)]\right)$ and pass to the limit!

If $V(x) \in \mathcal{C}^1[0,1]$ is increasing, $V(0) = 0$, and $V_k = V(k/N)$, then one can prove

"Theorem" 1: $\lim_{N \to \infty} \mathbb{E}[X(t)] = V^{-1}(t)$

"Theorem" 2: In the limit the dynamic model is (in a sense) static, and the limiting strategy is $d/dt(V^{-1})(t)dt$

Prezi

The Static Model:

Consider a N-player situation in which:

(N-1) players play $g_N(t) \in \mathcal{M}_1(\mathbb{R}_+)$

1 player play $\delta_x \in \mathcal{M}_1(\mathbb{R}_+)$ (quit at t=x)

Then $g_N(t)$ is a Nash-equilibrium (and ESS candidate) iff. the expected payoff of playing δ_x is constant w.r.t x.

$$\Rightarrow \begin{cases} \frac{dG_N}{dx} = \frac{1-G_N^{N-1}}{(N-1)\sum_{r=0}^{N-2} c_r \binom{N-2}{r} G_N^r (1-G_N)^{N-2-r}} =: \Xi(G_N) \\ G_N(0) = 0, \end{cases}$$

where $G_N(t)$ is the c.d.f of $g_N(t)$.

Note: $G_N(t)$ ⟷ $\mathbb{E}[X(t)]$

Theorem: Let $V(x)$ be an increasing C^1-function on [0,1] such that $V(0) = 0$ and define $\{V_k\}_{k=1}^N$ by $V_k := V(k/N)$. Then, if G_N is the unique solution to the ode-problem, it holds that

$$G_N(t) \longrightarrow \begin{cases} V^{-1}(t), & 0 \leq t \leq V(1) \\ 1, & t > V(1) \end{cases}$$

uniformly as $N \longrightarrow \infty$.

Intuition: The dynamic and the static models "coincides" in the limit!

Sketch of proof:

* $$\Xi_N(x) = \frac{1 - x^{N-1}}{(N-1)\sum_{r=0}^{N-2}(V_{r+2} - V_{r+1})\binom{N-2}{r}x^r(1-x)^{N-2-r}} \longrightarrow \frac{1}{V'(x)}$$

  uniformly in x.

* Thus, if $y_N(x)$ solves the N-player eq. and $y(x) = V^{-1}(x)$, we get the pointwise estimate:

$$|y_N(x) - y(x)| \leq \varepsilon_N x e^{xC_N}, \quad x \in [0, V(1)]$$

* Pointwise convergence $\Longrightarrow$ Uniform convergence since $\{y_N\}_{N=2}^{\infty}$ is a sequence of monotone functions.

Games Having a Continuum of Players

$\mathfrak{P} = (P, \mathcal{P}, \mu)$ - space of players

$\mathfrak{A} = (A, \mathcal{B}(A))$ - space of possible actions

A measure valued mapping $\Delta : P \to \mathcal{M}_1(A)$ (mixed action profile) keeping track of what strategies the players use ($\Delta(p)(A) = 1$).

$\mathcal{J} : \mathcal{R} \times P \to [-\infty, \infty)$ - payoff function ($\Delta \in \mathcal{R}$)

A GAME is a triple

$$\mathfrak{G} = (\mathfrak{P}, \mathfrak{A}, \mathcal{J})$$

In this frame work we can define what an ESS should be!

Assume a Continuum of Players playing the War of Attrition...

The payoff function is then given by:

$$\mathcal{J}(\Delta, p) := \int_0^\infty \left[V\left(\int_0^t \bar{\Delta}(dx) \right) - t \right] \Delta(p)(dt)$$

where $V(x)$ is an increasing $\mathcal{C}^2$-function on [0,1]. (prize function)
A calculation shows that the limit strategy $q(t) := d/dt(V^{-1}(t))$ is an ESS in the continuum limit of the static war of attrition if $V(x)$ is a CONVEX function. Moreover, for a CONCAVE prize function, the limit strategy $q(t)$ does worse than any other strategy.

IS THIS REFLECTED IN THE FINITE N-PLAYER GAME?

A sufficient condition for the N-player candidate strategy $G_N(t)$ to be an ESS is to have strict positivity in the function

$$Q\left[G_N\right] = 2G_N^{N-2} + \frac{d}{dt}\left\{\sum_{r=0}^{N-2} c_r \binom{N-2}{r} G_N^r (1-G_N)^{N-2-r}\right\}$$

Positive if N large enough and prize sequence is convex?

Theorem: If the prize sequence $\{V_k\}_{k=1}^{N} \subset \mathbb{R}_+$ is convex, then $G_N(t)$ is an ESS (unique) for ALL $N \geq 2$.

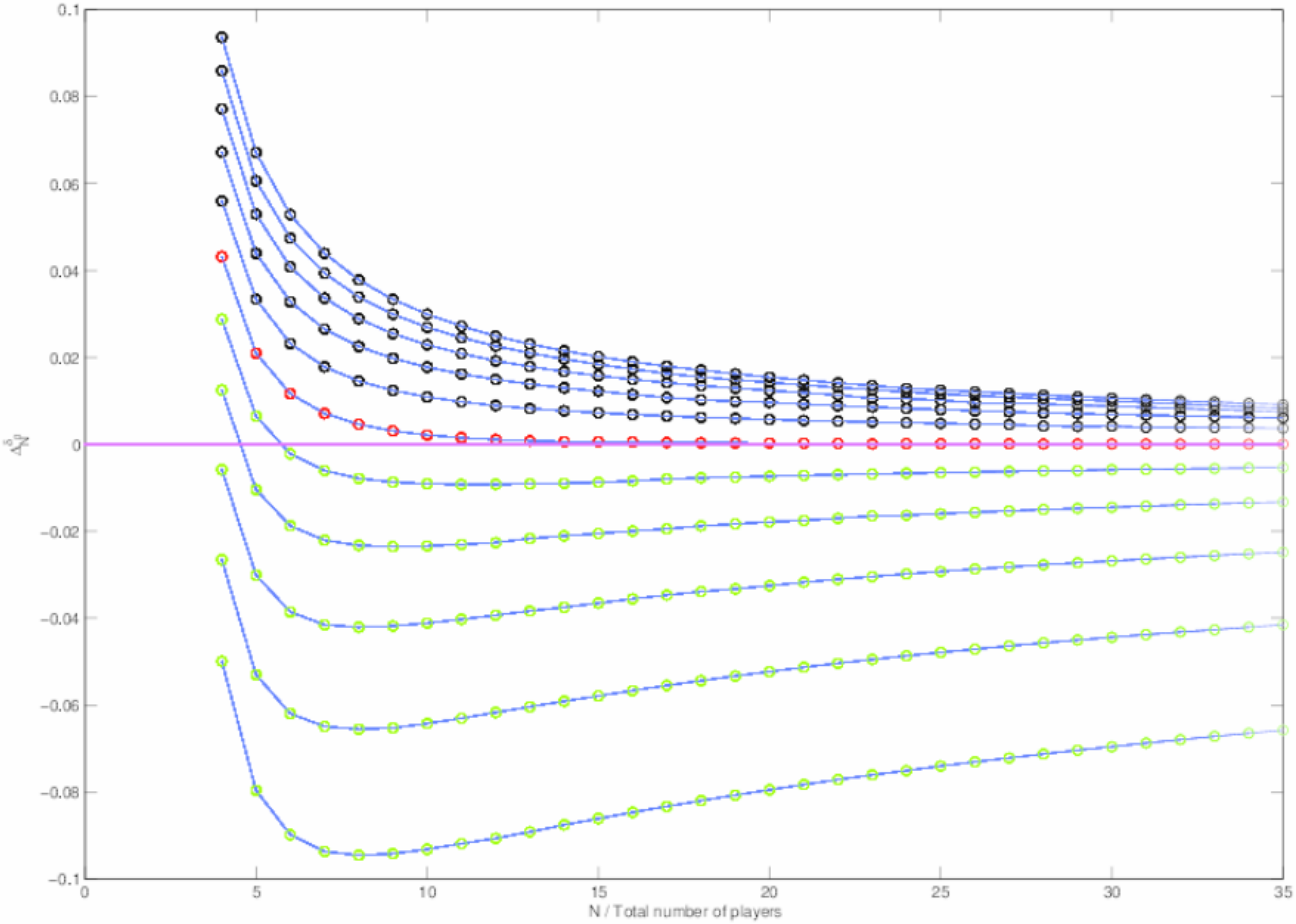
The Concave Case

The sufficient condition cannot be used in this case...

IDEA: pick a strategy and prove that $g_N(t)$ does not fulfill the (N-player) ESS conditions against this strategy.

EX: let δ_0 play against a population of $g_N(t)$-players.
If $\Delta_N^{\delta_0} := \mathcal{J}_N(g_N | g_N^{\oplus(N-2)}, \delta_0) - \mathcal{J}_N(\delta_0 | g_N^{\oplus(N-2)}, \delta_0) < 0$,
then $g_N(t)$ is not an ESS.

Hard to investigate $\Delta_N^{\delta_0}$ for a general prize sequence, but if we consider the case $V_k := (k/N)^\alpha$ so that the sequence is concave if $0 < \alpha < 1$, then $\Delta_N^{\delta_0}$ is negative for N large enough!


0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.1
0
5
10
15
20
25
30
35
N / Total number of players

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THANK YOU!