Stability of mean field model for opinion dynamics and collective motion

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Opinion dynamics

• Propose an opinion model for an interacting population of agents to study consensus convergence, with simple local rules of interaction.

• Opinion model for N agents [1]:

$$dx_i = -\frac{1}{N}\sum_{j=1}^{N}\phi(|x_i - x_j|)(x_i - x_j)dt + \sigma dW^i(t), \qquad i = 1, \ldots, N,$$

where $x_i(t)$ is the agent i's opinion.

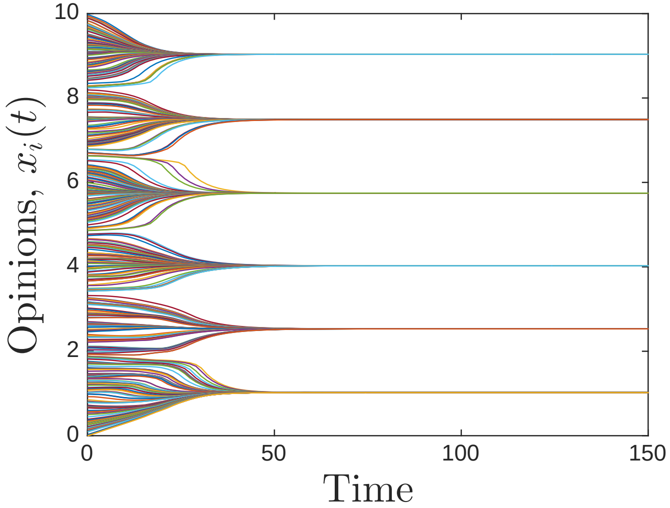
• The influence function ϕ is nonnegative, bounded, and compactly supported in $[0, 1]$.
$\hookrightarrow$ The interactions are attractive and the agent i is only affected by agents that have similar opinions.

• The initial opinions $x_i(0)$ may be deterministic or random, for instance $x_i(0)$ are i.i.d. with distribution with density ρ_0.

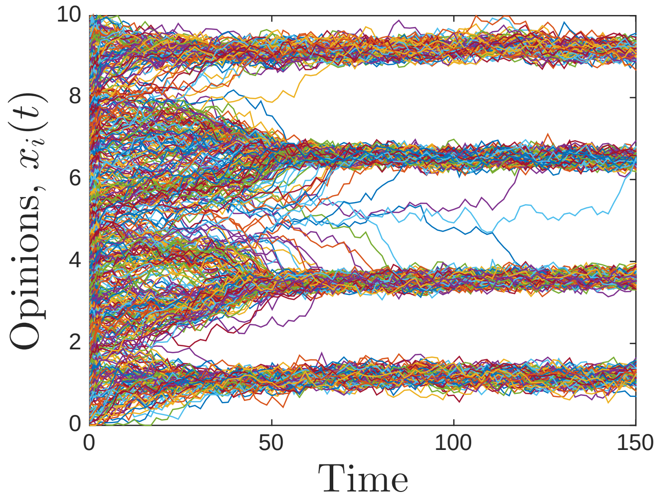
• The independent Brownian motions $W^i(t)$, $i = 1, \ldots, N$ model external noise, and $\sigma \geq 0$ is its strength.

[1] S. Motsch and E. Tadmor, *SIAM Review* **56**, 577 (2014).

Opinion dynamics: Simulations



$\sigma = 0$



$\sigma = 0.1$

$\phi(s) = \mathbf{1}_{[0,1/\sqrt{2}]}(s) + 0.1 \times \mathbf{1}_{(1/\sqrt{2},1]}(s)$.

Initial uniform distribution over $[0, L]$, $L = 10$, $N = 500$.

Can we predict the number of clusters ?

Opinion dynamics: The mean field limit

- Consider the model in $[0, L]$ with periodic boundary conditions.
- Introduce the empirical probability measure $\rho^N(t, dx)$ of the opinions of the agents:

$$\rho^N(t, dx) = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(t)}(dx).$$

$\rho^N(t, dx)$ is a measure-valued stochastic process.

- Assume that as $N \to \infty$, $\rho^N(0, dx)$ converges weakly, in probability, to a deterministic measure with density $\rho_0(x)$.

This happens (with $\rho_0(x) = 1/L$) if the initial opinions are i.i.d. with uniform density over $[0, L]$.

- As $N \to \infty$, $\rho^N(t, dx)$ converges weakly, in probability, to a deterministic probability measure whose density $\rho(t, x)$ satisfies (in a weak sense) the nonlinear Fokker-Planck equation [1]:

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \left\{ \left[\int \rho(t, x - y) y \phi(|y|) dy \right] \rho(t, x) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2}(t, x),$$

with initial density $\rho_0(x)$.

[1] D. A. Dawson, *J. Statist. Phys.* **31**, 29 (1983).

Opinion dynamics: The mean field limit

Formally:

$$dx_i = -\Big[\frac{1}{N}\sum_{j=1}^{N}\phi(|x_i - x_j|)(x_i - x_j)\Big]dt + \sigma dW^i(t), \qquad i = 1, \ldots, N$$

$$\Updownarrow$$

$$dx_i = -\Big[\int \phi(|x_i - y|)(x_i - y)\rho^N(t, dy)\Big]dt + \sigma dW^i(t), \qquad i = 1, \ldots, N$$

$$\downarrow N \to \infty$$

$$dX_t = -\Big[\int \phi(|X_t - y|)(X_t - y)\rho(t, y)dy\Big]dt + \sigma dW(t), \qquad \rho(t, \cdot) = \text{ pdf of } X_t$$

$$\Updownarrow$$

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x}\left\{\left[\int \phi(|x - y|)(x - y)\rho(t, y)dy\right]\rho(t, x)\right\} + \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2}(t, x)$$

[1] D. A. Dawson, *J. Statist. Phys.* **31**, 29 (1983).

Opinion dynamics for $\sigma = 0$

- Modulational instability (in the mean field limit $N = \infty$).
- Linearize the Fokker-Planck equation by assuming $\rho(t,x) = \rho_0 + \rho_1(t,x)$, $\rho_0 = 1/L$:

$$\frac{\partial \rho_1}{\partial t}(t,x) = \rho_0 \int \frac{\partial \rho_1}{\partial x}(t,x-y) y\phi(|y|)dy.$$

- Take the Fourier transform in x, $\hat{\rho}_1(t,k) = \int_0^L e^{-ikx}\rho_1(t,x)dx$, with the discrete frequency k in $\mathcal{K} = \{2\pi n/L, n \in \mathbb{N}\}$:

$$\frac{\partial \hat{\rho}_1}{\partial t}(t,k) = \left[i\rho_0 k \int e^{-iky} y\phi(|y|)dy \right] \hat{\rho}_1(t,k).$$

- For each k, $|\hat{\rho}_1(t,k)| = |\hat{\rho}_1(0,k)| \exp(\gamma_k t)$, where the growth rate of the k-th mode is:

$$\gamma_k = \mathrm{Re} \left[i\rho_0 k \int e^{-iky} y\phi(|y|)dy \right] = \rho_0 k \int \sin(ky) y\phi(|y|)dy.$$

- The growth rate γ_k is maximal for $k = \pm k_{\max}$ with

$$k_{\max} = \underset{k\in\mathcal{K}}{\arg\max} \left[\psi(k)\right], \qquad \psi(k) = 2k \int_0^1 \sin(ks)\phi(s)sds$$

Opinion dynamics for $\sigma = 0$

- Fluctuation theory (in the regime $N \gg 1$).
- Denote $\rho_0(dx) = dx/L$. Fix T. Consider

$$\rho_1^N(t,dx) := \sqrt{N}\left(\rho^N(t,dx) - \rho_0(dx)\right), \quad t \in [0,T]$$

- If the initial opinions $(x_j(0))_{j=1}^N$ are i.i.d. with uniform density over $[0,L]$, then, as $N \gg 1$,

$$\hat{\rho}_1^N(t=0,k) = \int e^{-ikx}\rho_1^N(t=0,dx) = \frac{1}{\sqrt{N}}\sum_{j=1}^N e^{-ikx_j(0)}, \quad k \in \{2\pi n/L, n \in \mathbb{N}^*\}$$

converge to $\hat{\rho}_1(t=0,k)$ i.i.d. complex circular Gaussian random variables, mean zero, and variance 1:

$$\mathbb{E}\left[\hat{\rho}_1(t=0,k)\right] = 0, \quad \mathbb{E}\left[\hat{\rho}_1(t=0,k)\overline{\hat{\rho}_1(t=0,k')}\right] = \delta_{kk'}, \quad k,k' \in \{2\pi n/L, n \in \mathbb{N}^*\}.$$

Note $\hat{\rho}_1^N(t,k=0) = 0$.

- As $N \gg 1$, $\hat{\rho}_1^N(t,k)$, $k \in \mathcal{K}\backslash\{0\}$, converge to $\hat{\rho}_1(t,k)$, $k \in \mathcal{K}\backslash\{0\}$, independent complex circular Gaussian random variables, with mean zero and variance $\exp(2\gamma_k t)$:

$$\mathbb{E}\left[\hat{\rho}_1(t,k)\overline{\hat{\rho}_1(t,k')}\right] = \delta_{kk'}\exp(2\gamma_k t), \quad k,k' \in \{2\pi n/L, n \in \mathbb{N}^*\}.$$

$\rightarrow$ The mode with the largest growth rate $\gamma_{\max} = 2\rho_0\psi(k_{\max})$ quickly dominates.

Opinion dynamics for $\sigma = 0$

- The time up to the onset of clustering is when $\rho_1 \sim \sqrt{N}\rho_0$:

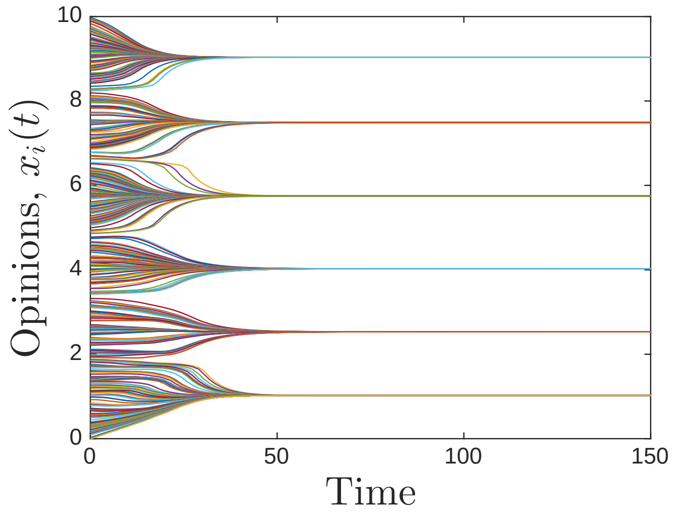
$$t_{clu} \simeq \frac{1}{2\gamma_{\max}} \ln N \simeq \frac{1}{2\rho_0 \psi(k_{\max})} \ln N$$

Clustering happens with a mean distance between clusters equal to $2\pi/k_{\max}$.

- Once clustering has occurred, two types of dynamical evolutions are possible:

1. If $2\pi/k_{\max} > 1$, then the clusters do not interact with each other because they are beyond the range of the influence function.
   $\hookrightarrow$ The situation is frozen and there is no consensus convergence.

Opinion dynamics for $\sigma = 0$



Here $\phi(s) = \mathbf{1}_{[0,1/\sqrt{2}]}(s) + 0.1 \times \mathbf{1}_{(1/\sqrt{2},1]}(s)$, $L = 10$, $N = 500$.

$k_{\max} = 3.77$. Inter-cluster distance $= 1.67$.

$\hookrightarrow$ No consensus convergence.

Opinion dynamics for $\sigma = 0$

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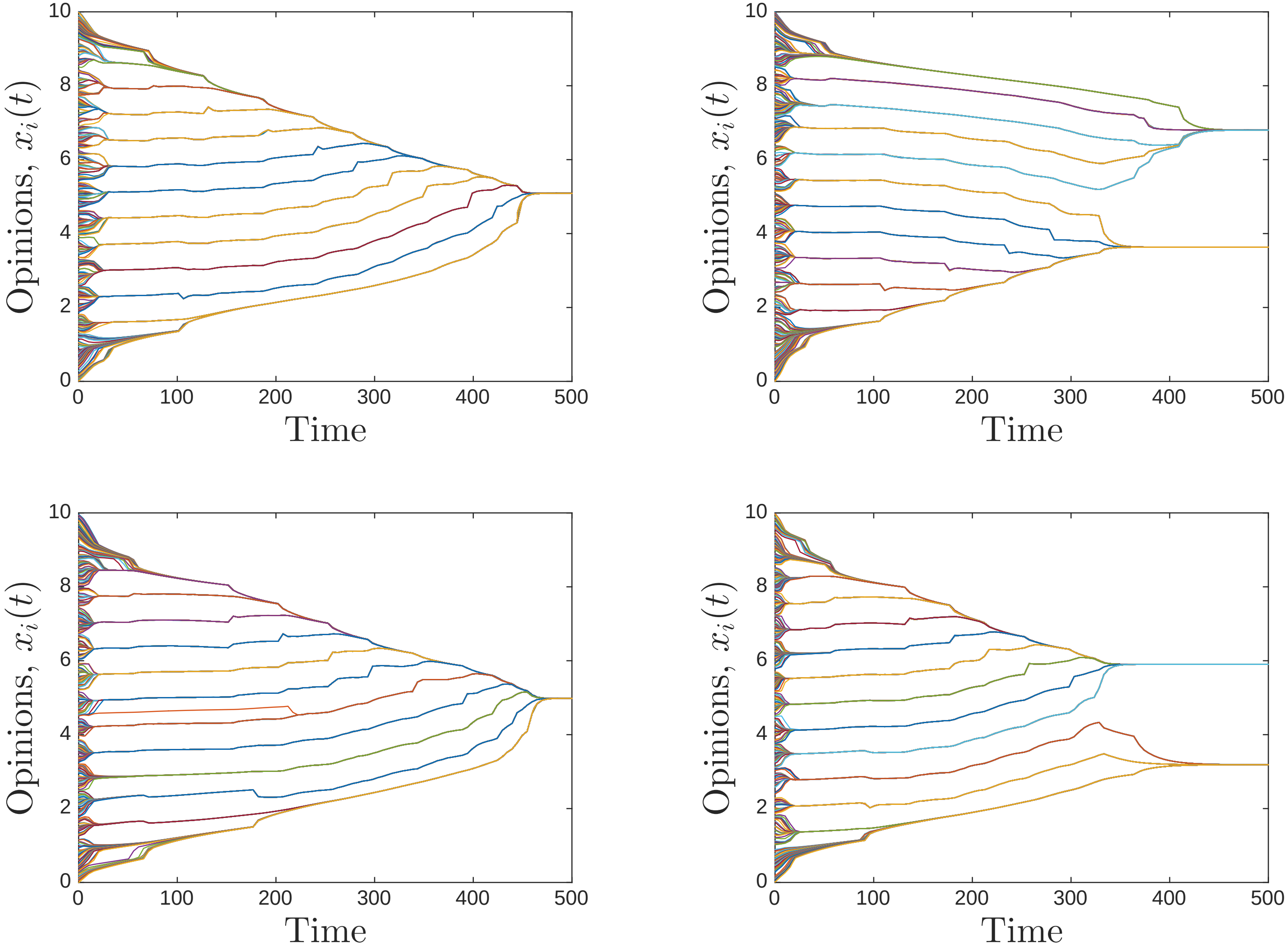
- Once clustering has occurred, two types of dynamical evolutions are possible:

1. If $2\pi/k_{\max} > 1$, then the clusters do not interact with each other because they are beyond the range of the influence function.
   $\hookrightarrow$ The situation is frozen and there is no consensus convergence.

2. If $2\pi/k_{\max} < 1$, then the clusters interact with each other.
   $\hookrightarrow$ There may be consensus convergence.
   However, consensus convergence is not guaranteed as clusters may merge by packets, and the centers of the new clusters may be separated by a distance larger than $2\pi/k_{\max}$, and then global consensus convergence does not happen.
   $\hookrightarrow$ The number of mega-clusters formed by this dynamic is not easy to predict.

Opinion dynamics for $\sigma = 0$



Here $\phi(s) = 0.2 \times \mathbf{1}_{[0,1/\sqrt{2}]}(s) + \mathbf{1}_{(1/\sqrt{2},1]}(s)$, $L = 10$, $N = 500$.

$k_{\max} = 9.42$. Inter-cluster distance $= 0.67$.

$\hookrightarrow$ There may be global consensus convergence (very sensitive !).

Opinion dynamics for $\sigma > 0$

- Modulational instability (in the mean field limit $N = \infty$).
- Linearize the Fokker-Planck equation by assuming $\rho(t,x) = \rho_0 + \rho_1(t,x)$, $\rho_0 = 1/L$:

$$\frac{\partial \rho_1}{\partial t}(t,x) = \rho_0 \int \frac{\partial \rho_1}{\partial x}(t,x-y) y \phi(|y|) dy + \frac{\sigma^2}{2} \frac{\partial^2 \rho_1}{\partial x^2}(t,x).$$

- In the Fourier domain:

$$\frac{\partial \hat{\rho}_1}{\partial t}(t,k) = \left[i\rho_0 k \int e^{-iky} y \phi(|y|) dy - \frac{\sigma^2 k^2}{2} \right] \hat{\rho}_1(t,k).$$

- Growth rates of the modes:

$$\gamma_{\sigma,k} = \mathrm{Re} \left[i\rho_0 k \int e^{-iky} y \phi(|y|) dy - \frac{\sigma^2 k^2}{2} \right],$$

or $\gamma_{\sigma,k} = \rho_0 \psi_\sigma(k)$, where

$$\psi_\sigma(k) = 2k \int_0^1 \sin(ky) y \phi(|y|) dy - \frac{\sigma^2 k^2}{2\rho_0}.$$

We look for the most unstable mode whose frequency $k_{\max} \in \{2\pi n/L, n \in \mathbb{N}\}$ maximizes $\psi_\sigma(k)$.

Opinion dynamics for $\sigma > 0$

Let

$$\sigma_c^2 = \max_{k \in \mathcal{K}} \left[\frac{4\rho_0}{k} \int_0^1 \phi(s) s \sin(ks) ds \right] \simeq 4\rho_0 \int_0^1 s^2 \phi(s) ds.$$

1. If $\sigma < \sigma_c$, then $\max_{k>0} \psi_\sigma(k) > 0$, so $k_{\max} = \text{argmax}_{k>0} \psi_\sigma(k) > 0$ exists and $\hat{\rho}_1(t, k_{\max})$ has positive growth rate $\gamma_{\max} = \rho_0 \psi_\sigma(k_{\max})$.
   $\hookrightarrow$ The system is linearly unstable (qualitatively analogous to the deterministic case, although $k_{\max}$ is reduced).

2. If $\sigma > \sigma_c$, then $\max_{k>0} \psi_\sigma(k) < 0$, so all of $\hat{\rho}_1(t, k)$ have negative growth rates.
   $\hookrightarrow$ The system with uniform density is stable.

Opinion dynamics for $\sigma > 0$

- Fluctuation theory (in the regime $N \gg 1$).
- The measure-valued process

$$\rho_1^N(t,dx) := \sqrt{N}\left(\rho^N(t,dx) - \rho_0(dx)\right)$$

converges in distribution as $N \to \infty$ to a measure-valued process $\rho_1(t,dx)$ whose density $\rho_1(t,x)$ satisfies a stochastic PDE:

$$d\rho_1(t,x) = \left[\rho_0 \int \frac{\partial \rho_1}{\partial x}(t,x-y)y\phi(|y|)dy + \frac{\sigma^2}{2}\frac{\partial^2 \rho_1}{\partial x^2}(t,x)\right]dt + \sigma dW(t,x)$$

where $W(t,x)$ is a space-time Gaussian random noise with mean zero and covariance

$$\mathrm{Cov}\left(\int_0^L W(s,x)f_1(x)dx, \int_0^L W(t,x)f_2(x)dx\right) = \frac{\min\{s,t\}}{L}\int_0^L f_1'(x)f_2'(x)dx$$

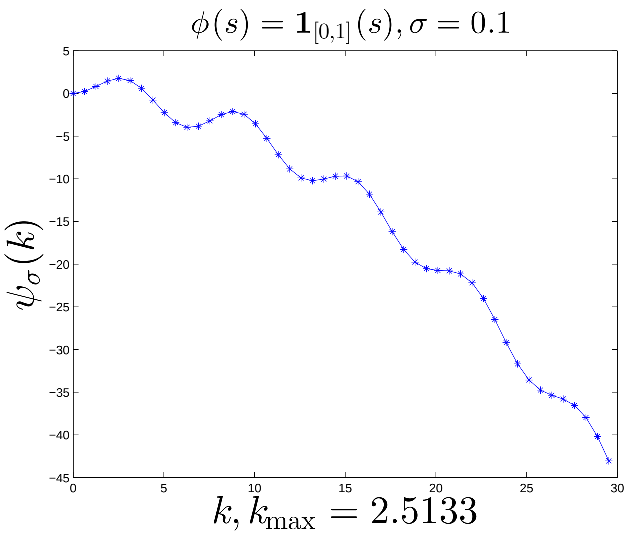
for any test functions $f_1(x)$ and $f_2(x)$, which is independent of the Gaussian (white noise) initial condition.

- $\hat{\rho}_1(t,k)$ is a Gaussian process with mean zero and covariance

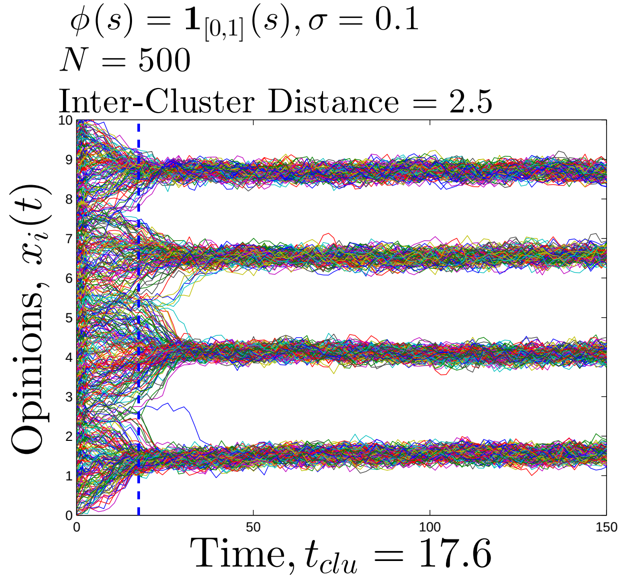
$$\mathbb{E}\left[\hat{\rho}_1(t,k)\overline{\hat{\rho}_1(t,k')}\right] = \delta_{kk'}\left\{\exp(2\gamma_{\sigma,k}t)\left[1+\frac{\sigma^2k^2}{2\gamma_{\sigma,k}}\right] - \frac{\sigma^2k^2}{2\gamma_{\sigma,k}}\right\}, \quad k,k' \in \{2\pi n/L, n \in \mathbb{N}\}.$$

$\to$ The mode with the largest growth rate $\gamma_{\sigma,k_{\max}} = \rho_0\psi_\sigma(k_{\max})$ quickly dominates.

Opinion dynamics for $\sigma > 0$

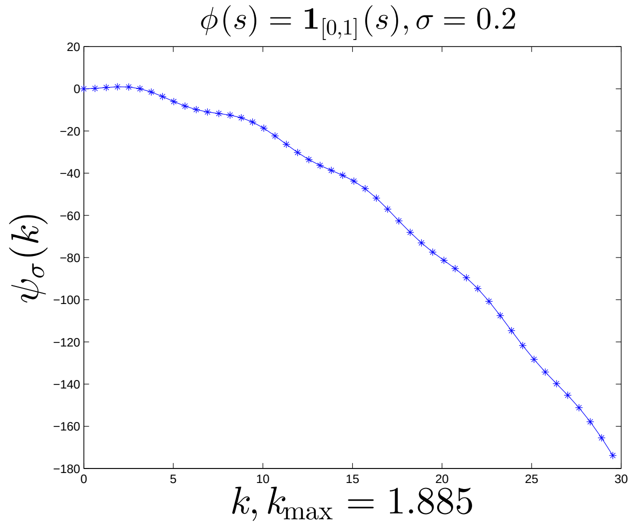


$\psi_\sigma(k)$

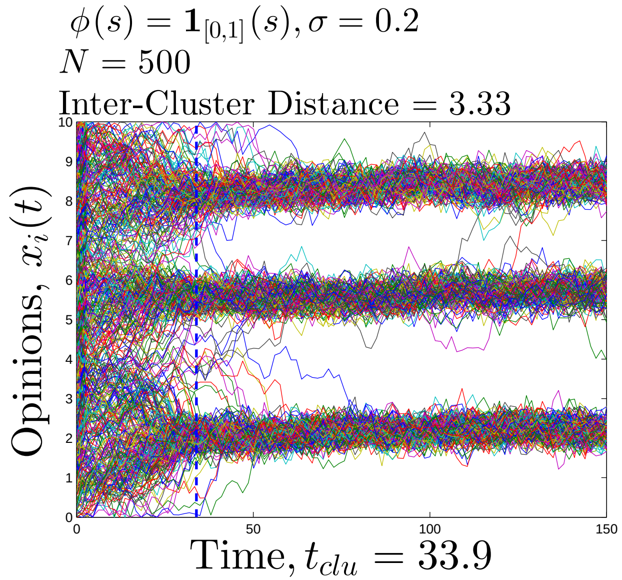


Simulation

Opinion dynamics for $\sigma > 0$

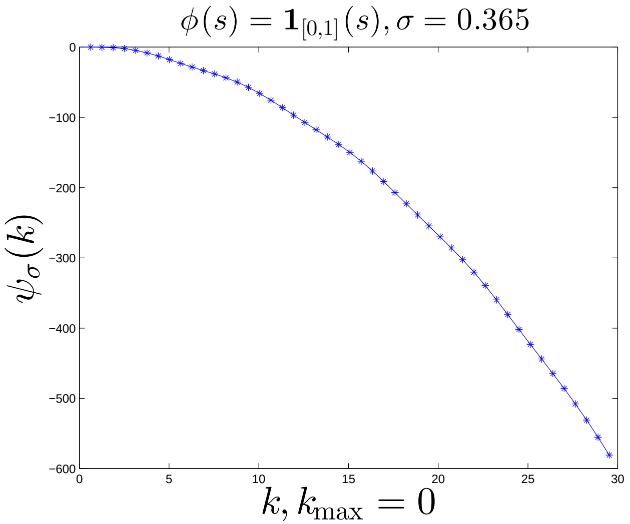


$\psi_\sigma(k)$

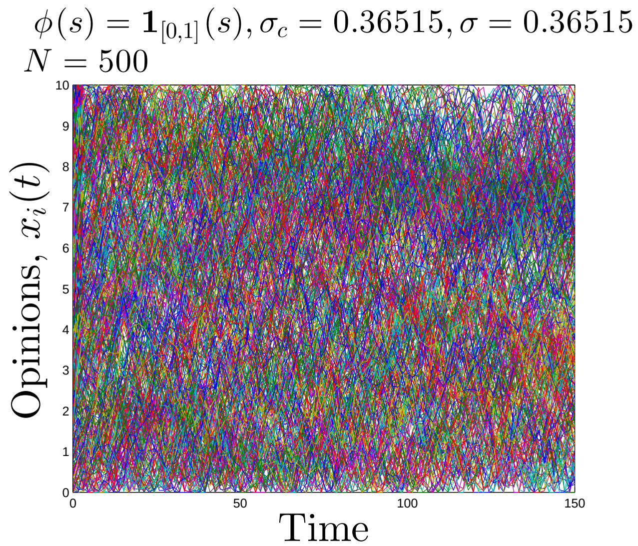


Simulation

Opinion dynamics for $\sigma > 0$

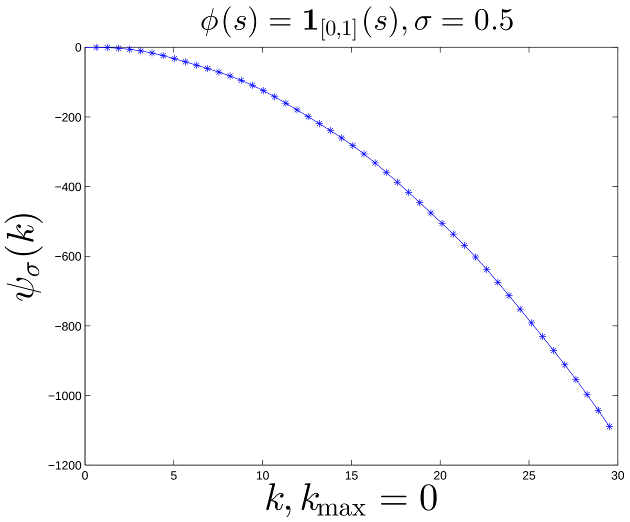


$\psi_\sigma(k)$

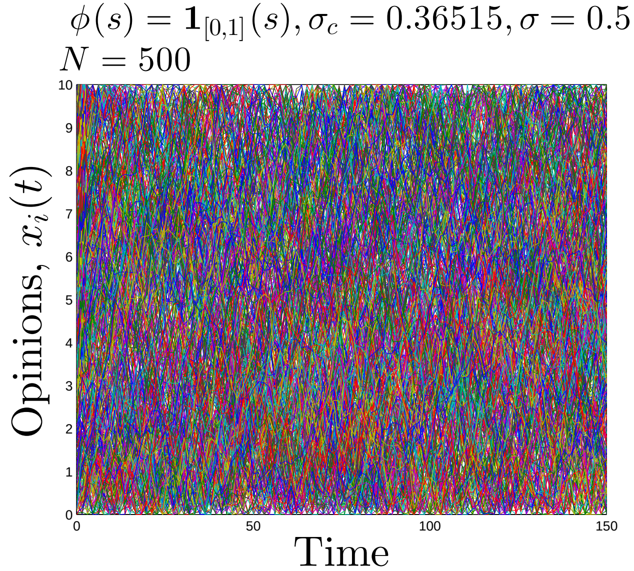


Simulation

Opinion dynamics for $\sigma > 0$



$\psi_\sigma(k)$

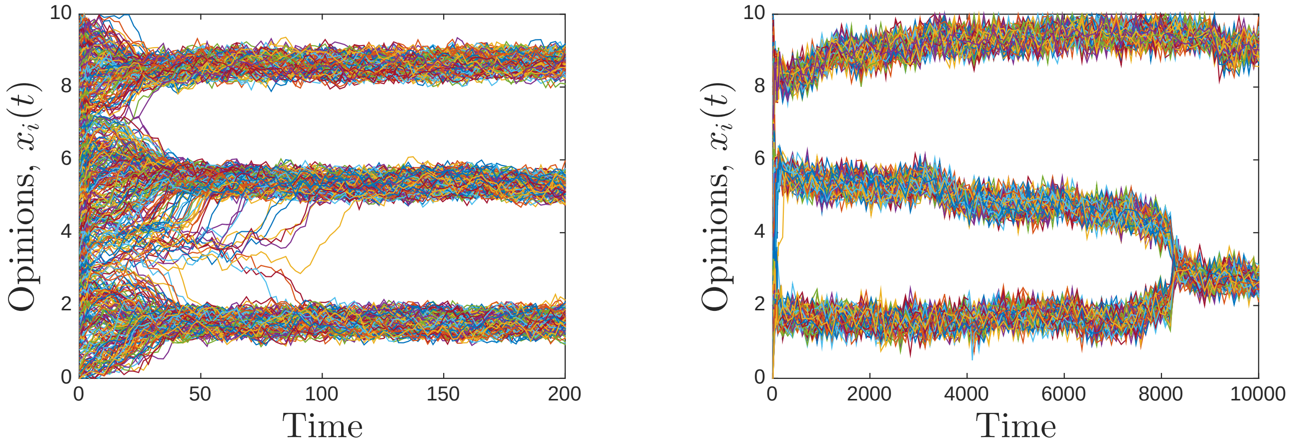


Simulation

Opinion dynamics for $\sigma > 0$

After clustering:

- Markovian dynamics of clusters (move like Brownian motions and merge when two of them come close to each other).



- Global consensus convergence (in 1D, Brownian motions always collide).

- Megacluster moves like a Brownian motion.

Collective motion

Collective motion: Czirók model

- N agents move along the torus $[0, L]$.
- For $i = 1, \ldots, N$, the position x_i and velocity u_i of particle i satisfy:

$$dx_i = u_i dt,$$
$$du_i = \left[G(\langle u \rangle_i) - u_i\right] dt + \sigma dW_i(t).$$

- $\{W_i(t)\}_{i=1}^N$ are independent Brownian motions.
- $\langle u \rangle_i$ is a weighted average of the velocities $\{u_j\}_{j=1}^N$:

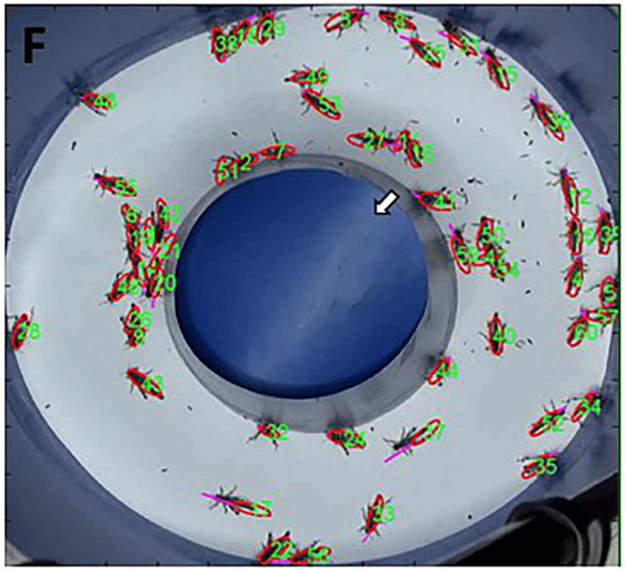
$$\langle u \rangle_i = \frac{1}{N} \sum_{j=1}^{N} u_j \phi(|x_j - x_i|),$$

with the weights depending on the distance (on the torus) between the position x_i and the positions of the other agents.
$\phi(x)$ is a nonnegative influence function normalized so that $\frac{1}{L} \int_0^L \phi(|x|) dx = 1$.

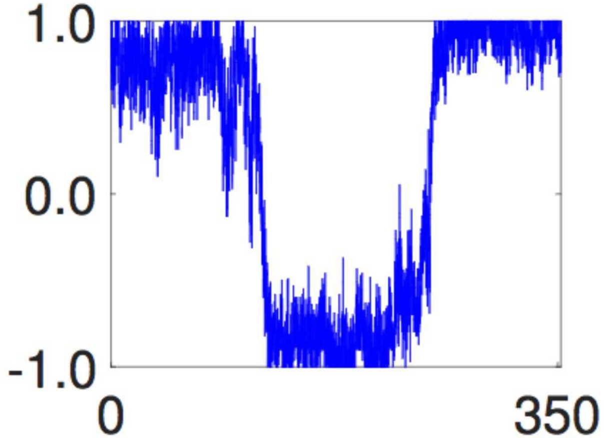
- $G(u)$ is an odd and smooth function.

If $G(u) = u$: Cucker-Smale model.
If $G(u)$ derives from a double well potential: Czirók model.

Collective motion bistability: Experiments



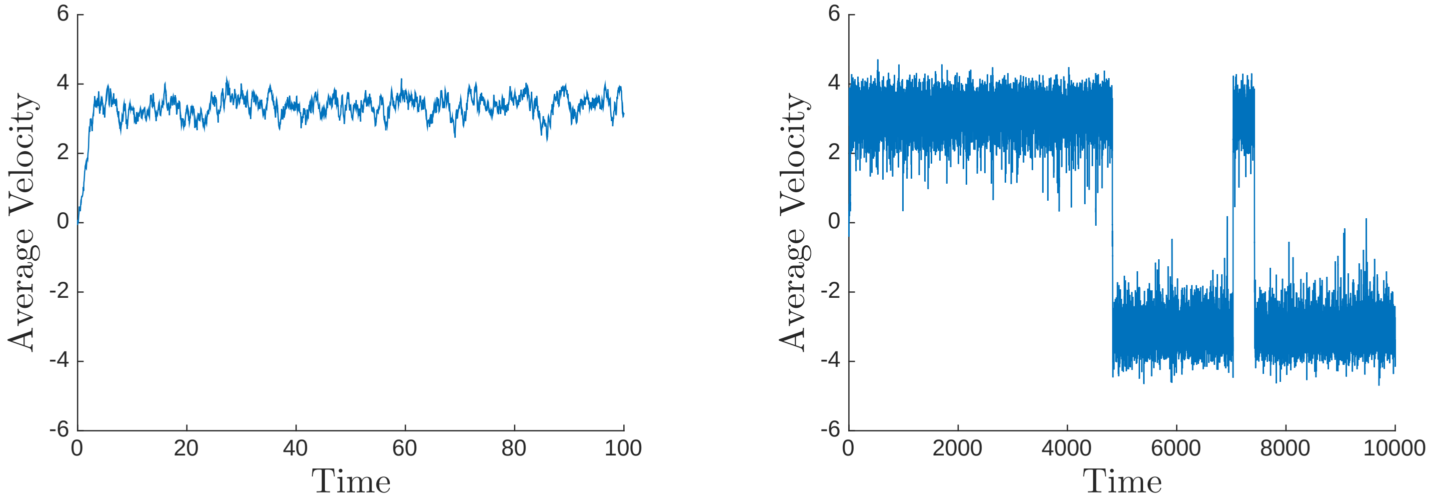
Locusts in a torus



Alignment as function of time

Cf: G. Ariel and A. Ayali, PLoS Comput. Biol. 11(12):1–25, 2015.

Collective motion: Simulations



$\phi(s) = 5 \times \mathbf{1}_{[0,1]}(s)$.

$G(u) = 2u - 0.072u^3$.

$\sigma = 5$.

Initial distribution: $(x_i(0), u_i(0))_{i=1}^N$ i.i.d., $(x_i(0))_{i=1}^N$ uniform over $[0, L]$, $(u_i(0))_{i=1}^N$ Gaussian distributed with mean zero and variance $\sigma^2/2$, $L = 10$, $N = 100$.

Can we explain the bistability, the transition rate ?

Collective motion: The mean field limit

• Introduce the empirical probability measure:

$$\rho_N(t, dx, du) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), u_i(t))}(dx, du).$$

• If $\rho_N(0, dx, du)$ converges to a deterministic measure $\bar{\rho}(x, u)dxdu$ as $N \to \infty$, then as $N \to \infty$, $\rho_N(t, dx, du)$ converges to the deterministic measure $\rho(t, x, u)dxdu$ whose density is the solution of the nonlinear Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial u} \left\{ \left[G \left(\iint u' \phi(|x'|) \rho(t, x - x', u') du' dx' \right) - u \right] \rho \right\} + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho}{\partial u^2},$$

starting from $\rho(t = 0, x, u) = \bar{\rho}(x, u)$.

Collective motion: The stationary states in the mean field limit

- The stationary solutions $\rho(x,u)$ have the following form:

$$\rho_\xi(x,u) = \frac{1}{L} F_\xi(u), \quad F_\xi(u) = \frac{1}{\sqrt{\pi\sigma^2}} \exp\Big(- \frac{(u-\xi)^2}{\sigma^2} \Big).$$

- They are uniform in space, Gaussian in velocity, and their mean velocity ξ satisfies the compatibility condition:

$$\xi = G(\xi).$$

- There are, therefore, as many stationary equilibria as there are solutions to the compatibility equation.

- When G is such that $u - G(u)$ derives from a double-well potential, such as $G(u) = 2\tanh(u)$ or $G(u) = 2u - u^3$, there are three ξ satisfying the compatibility condition: 0 and $\pm\xi_e$, with $\xi_e > 0$.

Collective motion: Linear stability analysis for the stationary states

- Let ξ be such that $G(\xi) = \xi$ and consider

$$\rho(t,x,u) = \rho_\xi(x,u) + \rho^{(1)}(t,x,u) = \frac{1}{L}F_\xi(u) + \rho^{(1)}(t,x,u),$$

for small perturbation $\rho^{(1)}$. By linearizing the nonlinear Fokker-Planck equation:

$$\begin{aligned}\frac{\partial \rho^{(1)}}{\partial t} &= -u\frac{\partial \rho^{(1)}}{\partial x} - \frac{\partial}{\partial u}\left[(\xi - u)\rho^{(1)}\right] \\ &\quad - \frac{1}{L}G'(\xi)\left[\iint u'\phi(|x'|)\rho^{(1)}(t,x-x',u')du'dx'\right]F_\xi'(u) + \frac{1}{2}\sigma^2\frac{\partial^2\rho^{(1)}}{\partial u^2}.\end{aligned}$$

- The mode $\rho_k^{(1)}(t,u) = \frac{1}{L}\int_0^L \rho^{(1)}(t,x,u)e^{i2\pi kx/L}dx$ satisfies

$$\begin{aligned}\frac{\partial \rho_k^{(1)}}{\partial t} &= \frac{i2\pi k}{L}u\rho_k^{(1)} - \frac{\partial}{\partial u}\left[(\xi - u)\rho_k^{(1)}\right] \\ &\quad - G'(\xi)\phi_k\left[\int u'\rho_k^{(1)}(t,u')du'\right]F_\xi'(u) + \frac{1}{2}\sigma^2\frac{\partial^2\rho_k^{(1)}}{\partial u^2},\end{aligned}$$

with $\phi_k = \frac{1}{L}\int_0^L \phi(|x|)e^{i2\pi kx/L}dx$.

- The equations are uncoupled in k and the system is linearly stable if all modes are stable.

Collective motion: Linear stability analysis for the stationary states

- The 0-th order mode is stable if and only if $G'(\xi) < 1$.
- The k-th order modes are stable if σ is large enough (critical σ depends on ϕ, G).

When G is such that $u - G(u)$ derives from a double-well potential:

1. the order states

$$\rho_{\pm\xi_e}(x,u) = \frac{1}{L}\frac{1}{\sqrt{\pi\sigma^2}}\exp\Big(-\frac{(u\mp\xi_e)^2}{\sigma^2}\Big), \quad \xi_e = G(\xi_e) > 0,$$

   are stable equilibria,

2. the disorder state

$$\rho_0(x,u) = \frac{1}{L}\frac{1}{\sqrt{\pi\sigma^2}}\exp\Big(-\frac{u^2}{\sigma^2}\Big)$$

   is an unstable equilibrium.

$\hookrightarrow$ The noise strength σ improves the stability of the order states $\rho_{\pm\xi_e}(x,u)$.

Collective motion: Fluctuation analysis

- Fluctuation analysis ($N \gg 1$).

- Let $\rho_N(t,dx,du) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i(t),u_i(t))}(dx,du)$ and ξ be a solution to $\xi = G(\xi)$.
- If as $N\to\infty$, $\rho_N(0,dx,du)$ converges to the stationary state $\rho_\xi(x,u)dxdu$,

then as $N\to\infty$, $\rho_N^{(1)}(t,dx,du) = \sqrt{N}[\rho_N(t,dx,du) - \rho_\xi(x,u)dxdu]$ converges to the measure-valued process $\rho^{(1)}(t,dx,du)$ satisfying

$$
\begin{aligned}
d\rho^{(1)} = &-u\frac{\partial \rho^{(1)}}{\partial x}dt - \frac{\partial}{\partial u}[(G(\xi)-u)\rho^{(1)}]dt \\
&- G'(\xi)\frac{\partial \rho_\xi}{\partial u}\left[\int_0^L\int_{-\infty}^{\infty} u'\phi(|x'-x|)\rho^{(1)}(t,dx',du')\right]dt + \frac{1}{2}\sigma^2\frac{\partial^2\rho^{(1)}}{\partial u^2}dt + \sigma dW_\xi,
\end{aligned}
$$

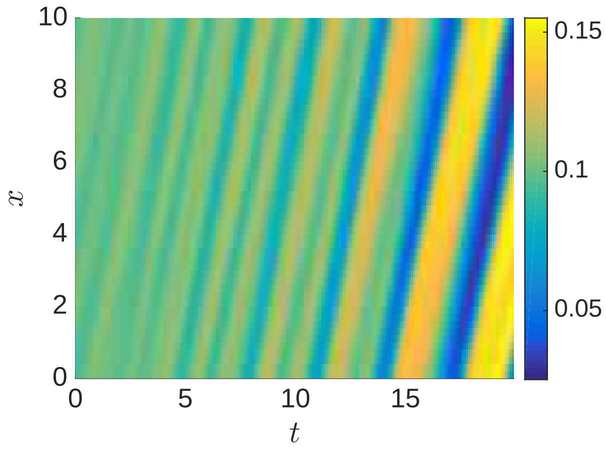
where $W_\xi(t,x,u)$ is a space-time Gaussian random noise with covariance

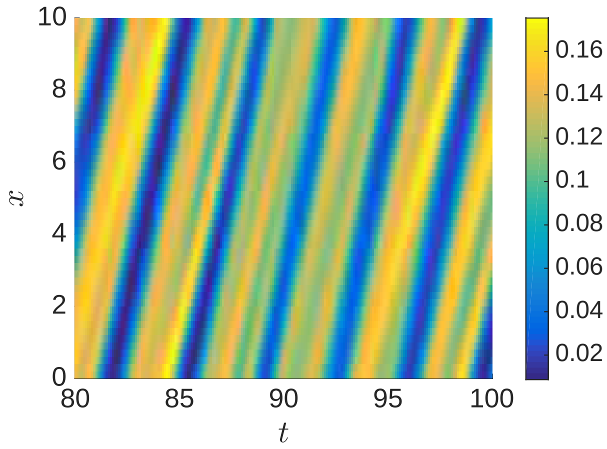
$$
\begin{aligned}
&\mathrm{Cov}\left(\int_0^L\int_{-\infty}^{\infty} W_\xi(t,x,u)f_1(x,u)dxdu, \int_0^L\int_{-\infty}^{\infty} W_\xi(t',x,u)f_2(x,u)dxdu\right) \\
&\quad = \min(t,t')\int_0^L\int_{-\infty}^{\infty}\frac{\partial f_1}{\partial u}(x,u)\frac{\partial f_2}{\partial u}(x,u)\rho_\xi(x,u)dxdu,
\end{aligned}
$$

for any test functions f_1 and f_2.

- Stability is ensured iff ρ_ξ is stable for the linearized Fokker-Planck equation.

Collective motion: Simulations





$G(u) = \frac{h+1}{5} u - \frac{h}{125} u^3$, $h = 6$, $\sigma = 0.5$, $N = 1000$.

- Here $G'(\xi) < 1$, so that the mode 0 is stable $\rightarrow$ the average velocity is stable.

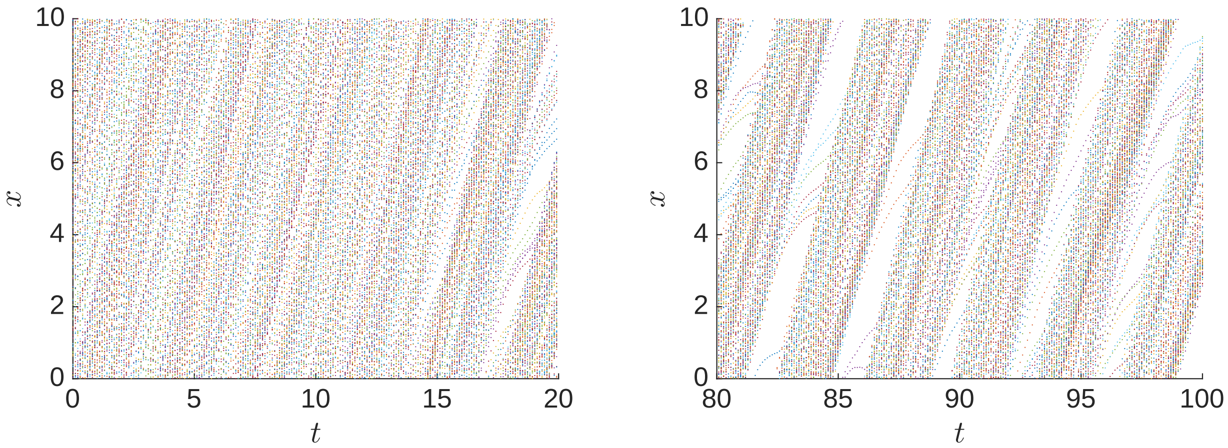
- The first mode $k_{\max} = 1$, with growth rate $\gamma(k_{\max}) = \gamma_{\rm r}(k_{\max}) + i\gamma_{\rm i}(k_{\max}) \in \mathbb{C}$, is the most unstable.

$\rightarrow$ a spatial modulation is growing, which gives one cluster.

- The imaginary part of the growth rate gives the velocity of the cluster $v = \frac{\gamma_{\rm i}(k_{\max})L}{2\pi k_{\max}}$:

$$\exp[-i2\pi k_{\max} x/L] \exp[(\gamma_{\rm r}(k_{\max}) + i\gamma_{\rm i}(k_{\max}))t] = \exp[-i(2\pi k_{\max}/L)(x - vt)] \exp[\gamma_{\rm r}(k_{\max})t]$$

Collective motion: Simulations



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- The velocity of the cluster is larger than the average velocity of the particles.

The slow particles of the cluster are left behind !

Collective motion: Large deviations analysis

- Assume that two order states $\rho_{\pm\xi_e}$ exist and are stable.
- Assume initial conditions are such that $\rho(0,x,u) \simeq \rho_{+\xi_e}(x,u)$.
- Consider the rare event:

$$A = \{\rho \, : \, \|\rho(T,dx,dy) - \rho_{-\xi_e}(x,u)dxdu\| \leq \delta\} \, .$$

- We have

$$\mathbf{P}(\rho_N \in A) \overset{N \gg 1}{\approx} \exp\left(-N \inf_{\rho \in A} I(\rho)\right) .$$

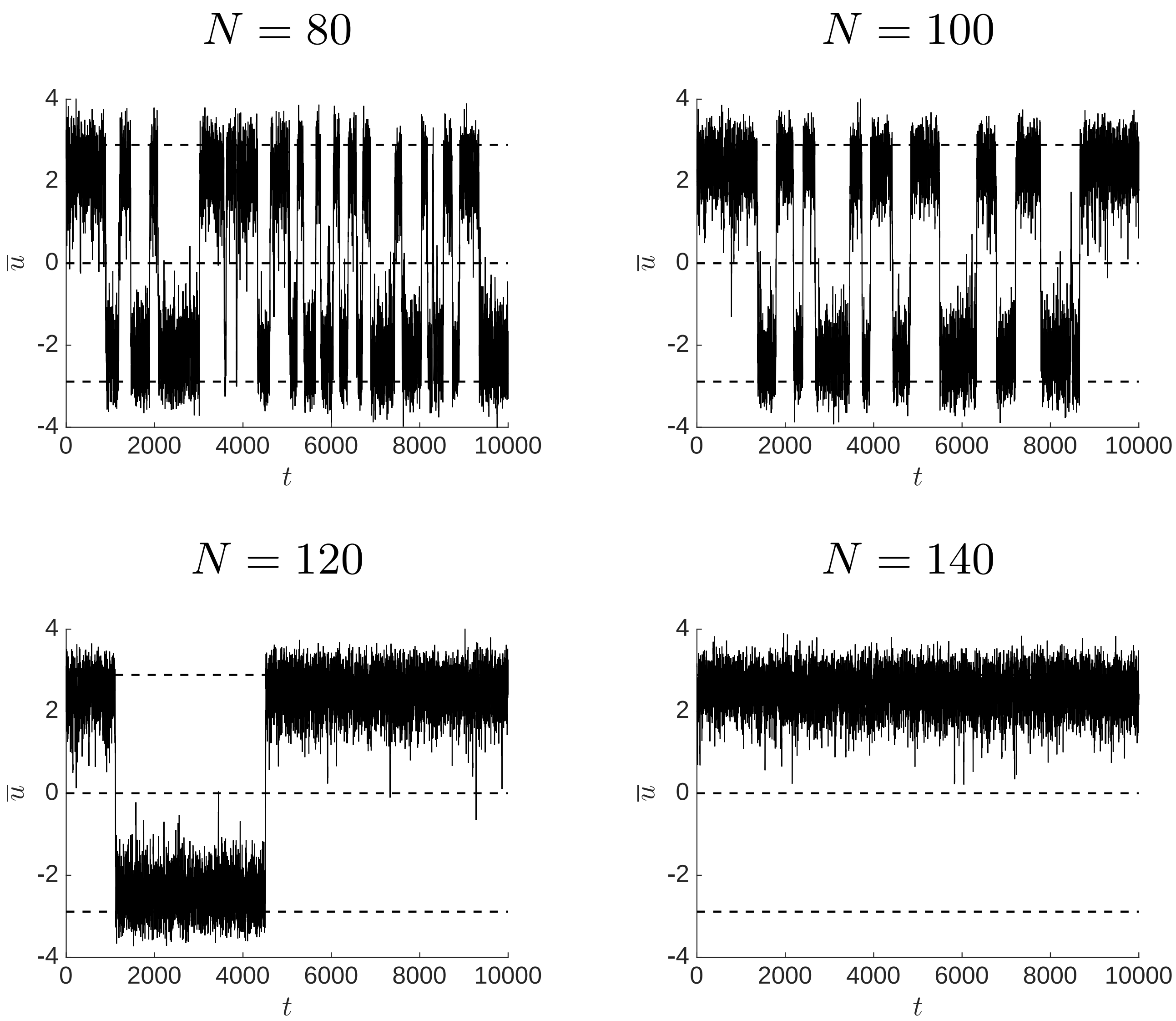
The rate function:

$$I(\rho) = \frac{1}{2\sigma^2} \int_0^T \sup_{f(x,u) : \langle \rho(t,\cdot,\cdot), (\frac{\partial f}{\partial u})^2 \rangle \neq 0} \frac{\langle \frac{\partial \rho}{\partial t}(t,\cdot,\cdot) - \mathcal{L}^*_{\rho(t,\cdot,\cdot)}\rho(t,\cdot,\cdot), f \rangle^2}{\langle \rho(t,\cdot,\cdot), (\frac{\partial f}{\partial u})^2 \rangle} dt,$$

where $\mathcal{L}^*_\nu$ is the differential operator associated to the Fokker-Planck equation:

$$\mathcal{L}^*_\nu \rho = -u \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial u}\left[\left(G\left(\iint u' \phi(|x'|)\nu(x-x',u')du'dx'\right) - u\right)\rho\right] + \frac{1}{2}\sigma^2 \frac{\partial^2 \rho}{\partial u^2}.$$

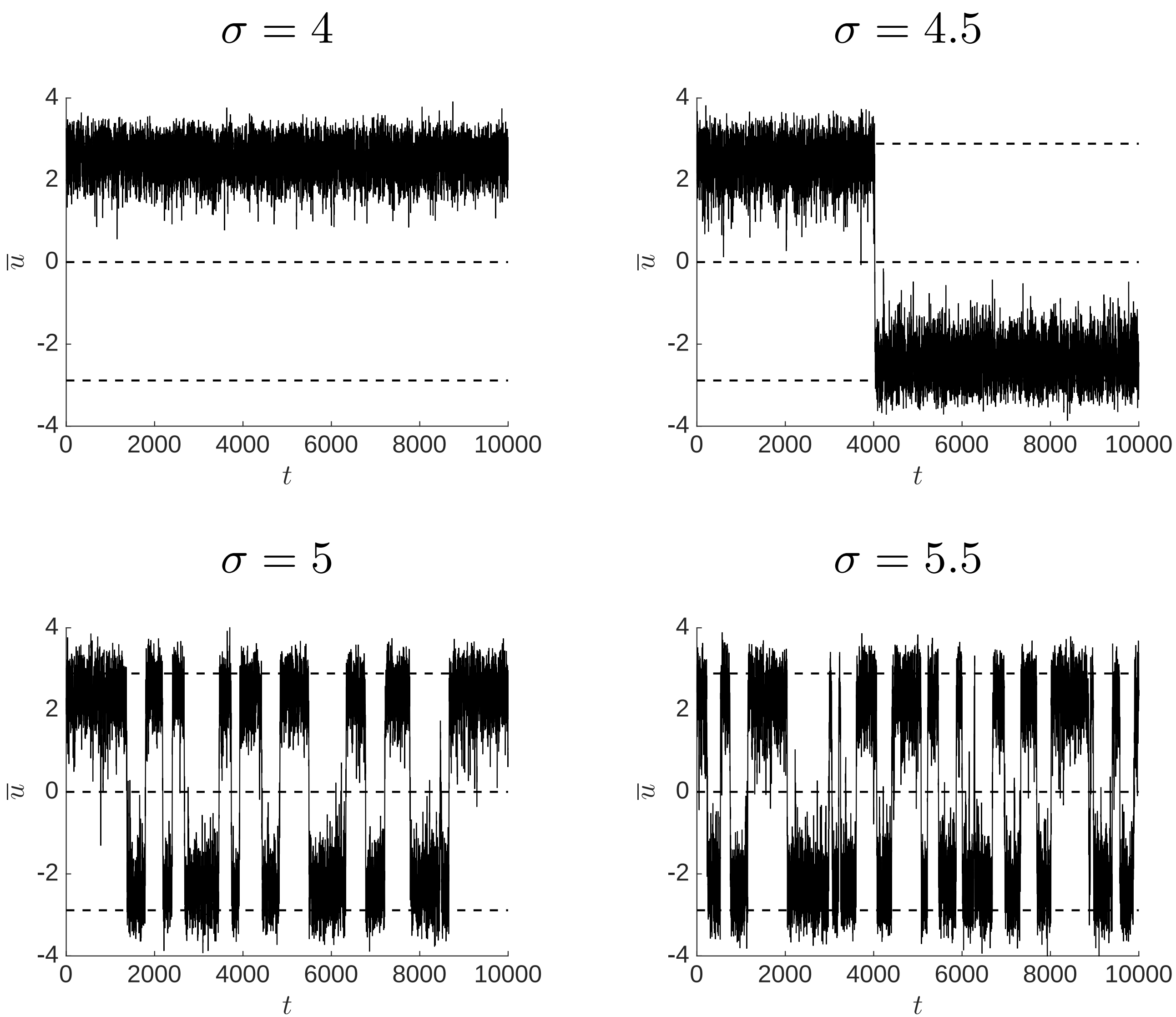
Collective motion: Simulations



Empirical average velocity $\bar{u}^n$. Here $h = 6$ and $\sigma = 5$, with $G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$.

Stability increases with N.

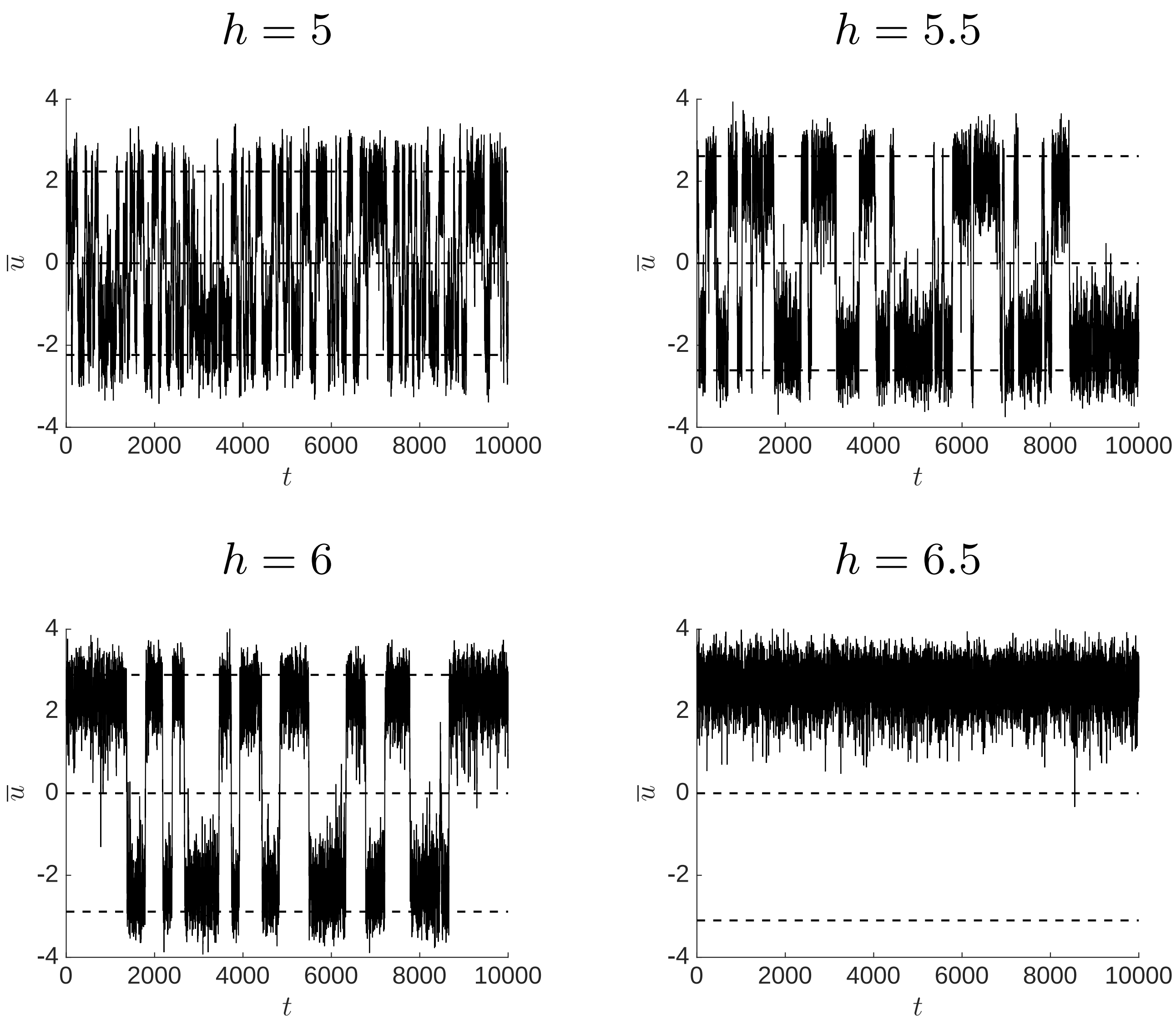
Collective motion: Simulations



Empirical average velocity $\bar{u}_n$. Here $N = 100$ and $h = 6$, with $G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$

Stability decreases with σ (but provided σ is large enough !).

Collective motion: Simulations



Empirical average velocity $\bar{u}_n$. Here $N = 100$ and $\sigma = 5$, with $G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$.

Stability increases with h.

Conclusions

- Linear stability analysis of the nonlinear Fokker-Planck equation of the mean field model gives a lot of insight into the dynamics of the interacting system.
- Noise globally increases the stability (for opinion dynamics and collective motion).
- Simple systems can give complex behaviors.