Invariants and representation spaces for shapes and forms

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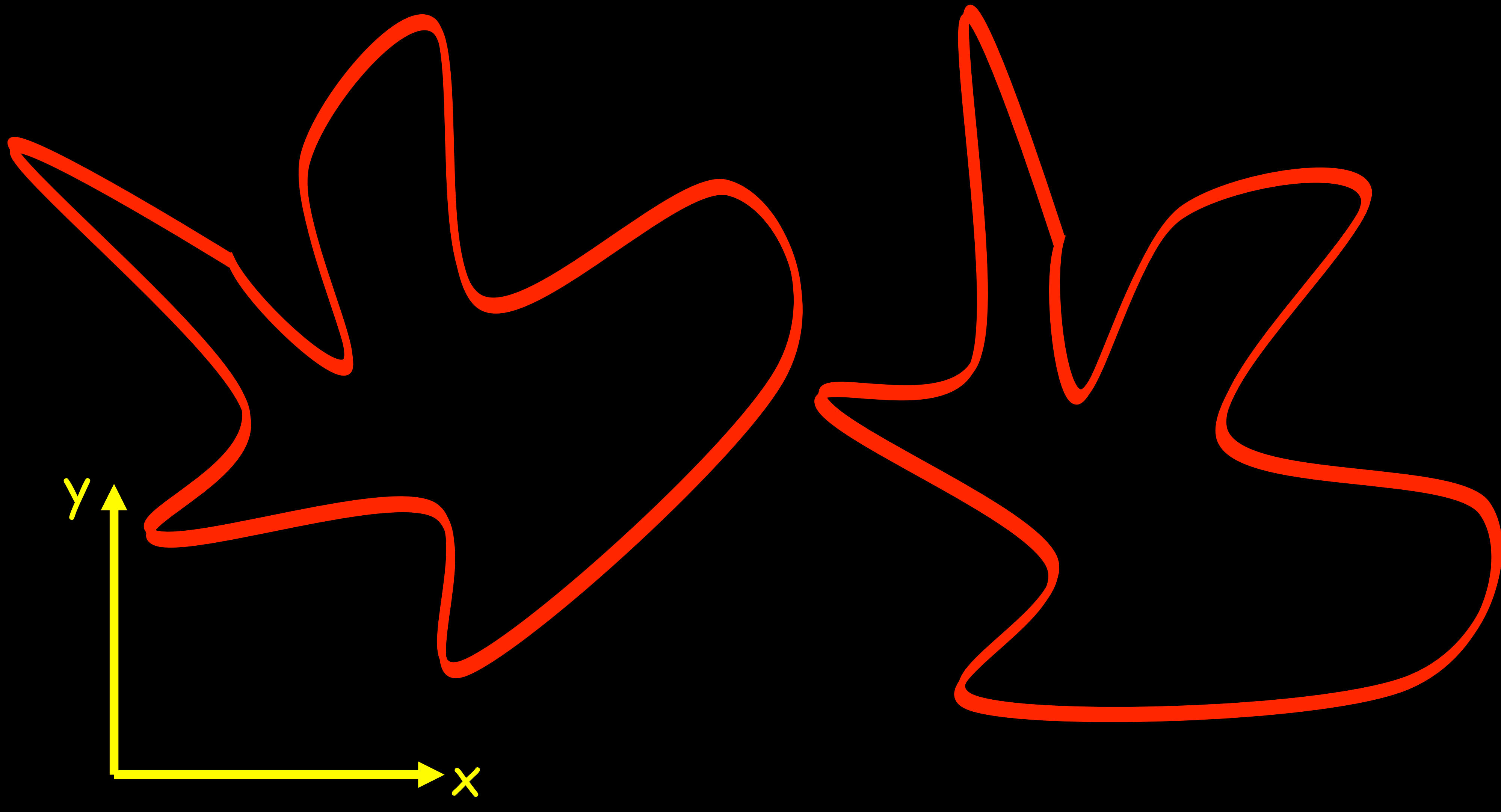
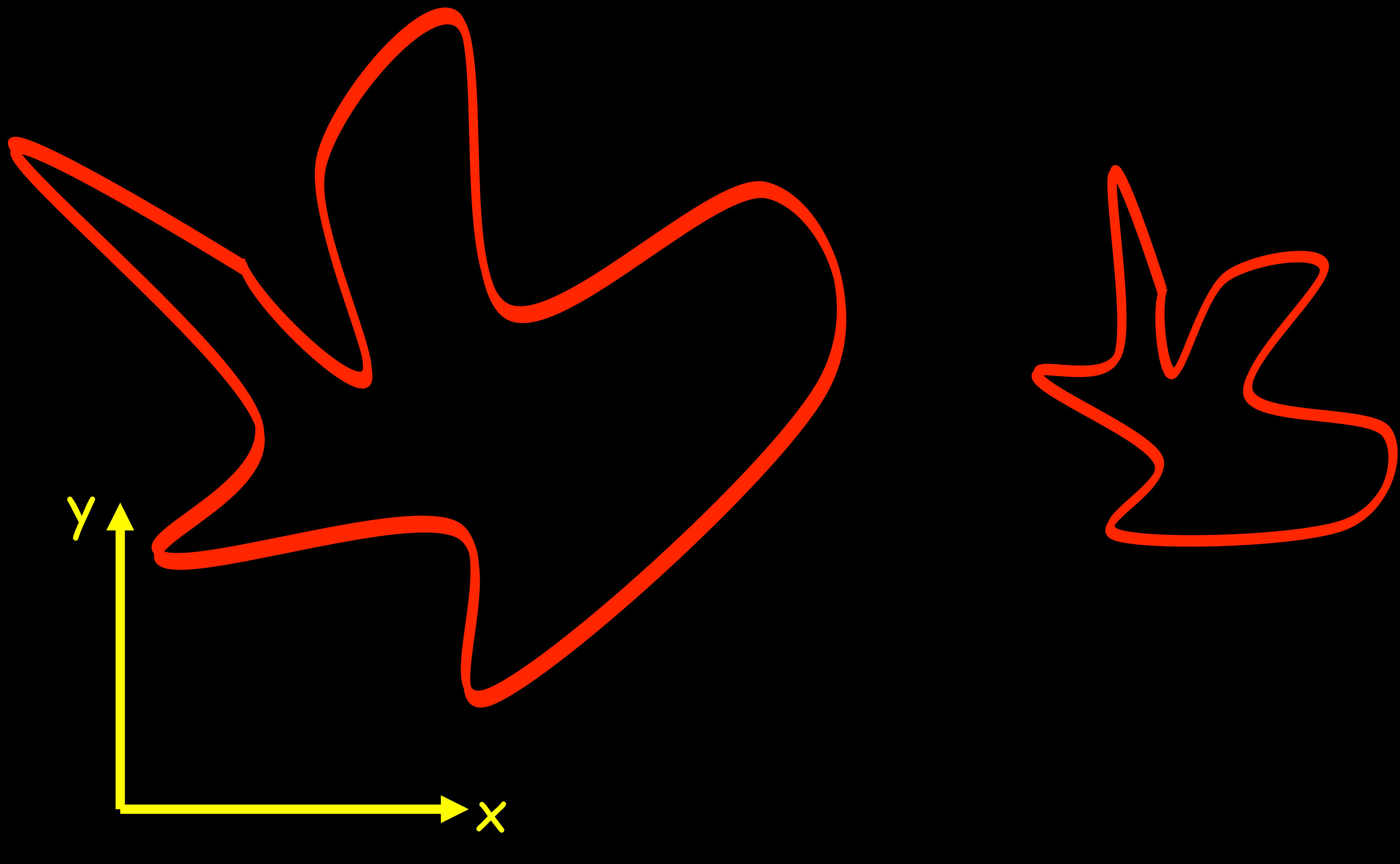
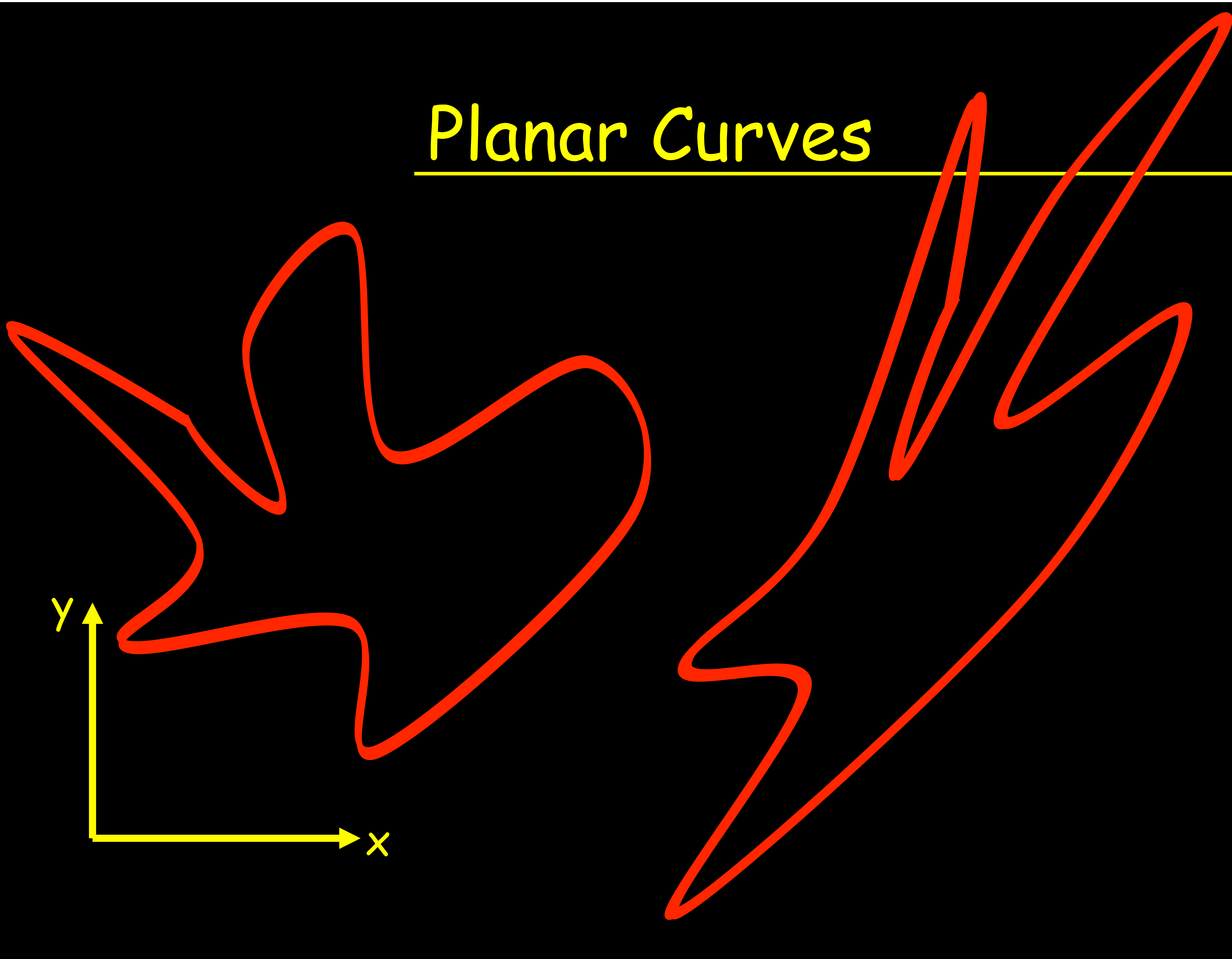
Planar Curves



Planar Curves

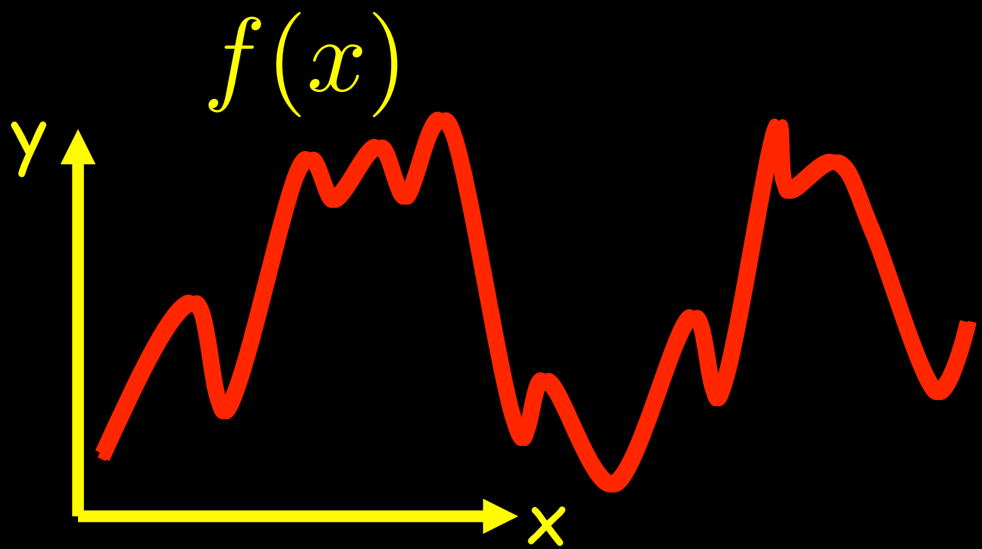


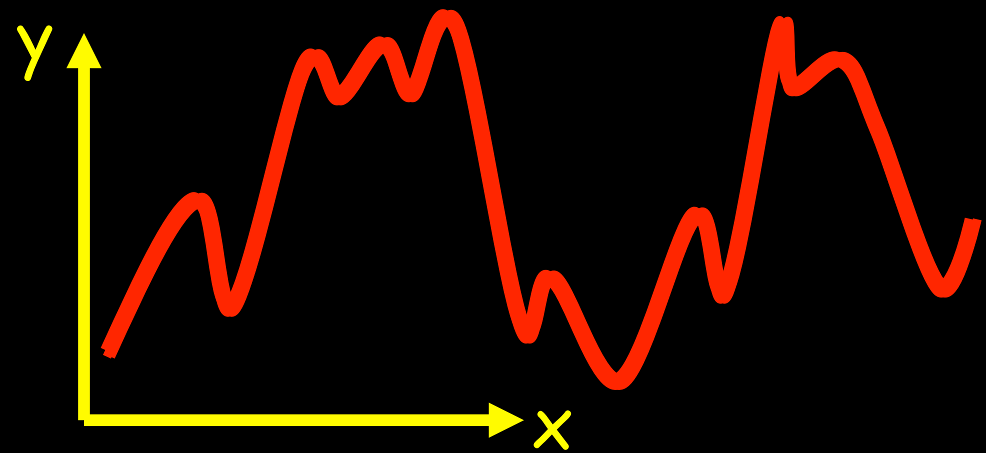
Planar Curves



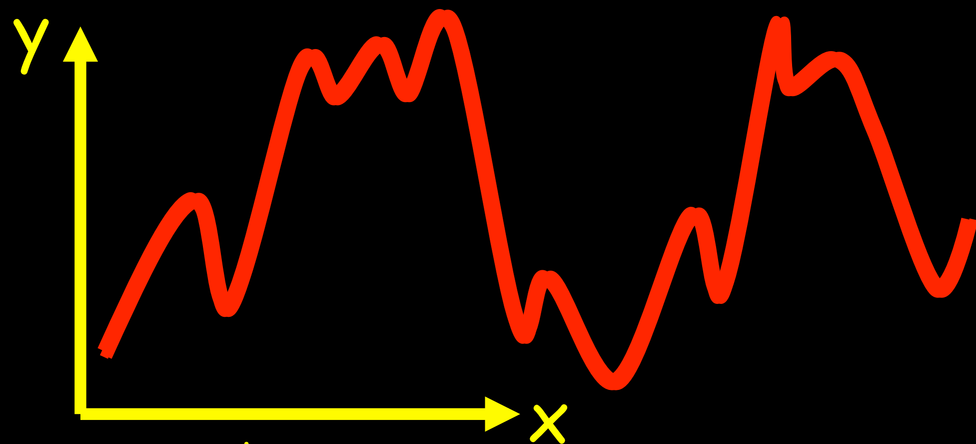
Planar Curves

Signals





Signal representation



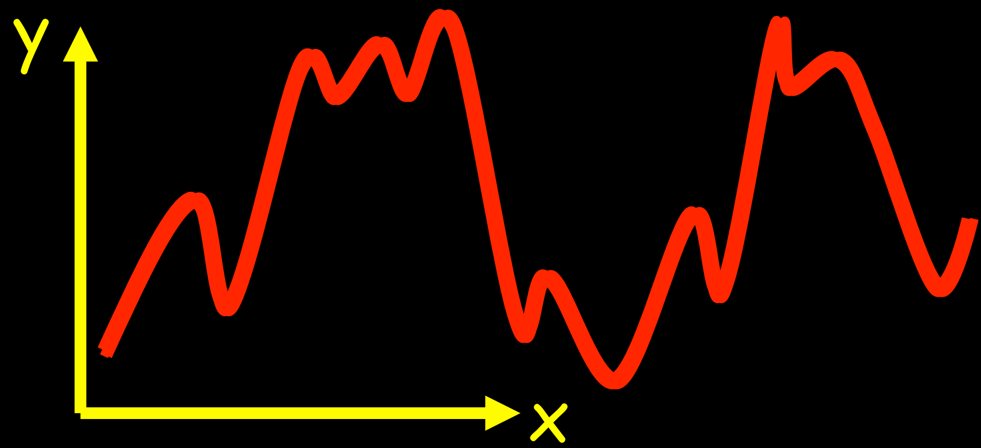
$$r_k = \left| f - \sum_{j=1}^{k} \langle f, \psi_j \rangle \psi_j \right|^2$$

$$\min_{\{\psi_j\}_{j=1}^{n}} \max_{|\nabla f|^2 \leq 1} r_k$$

Aflalo, Brezis, Bruckstein, K., Sochen

Signal representation

$$f \in \{F_1, ..., F_N\}$$

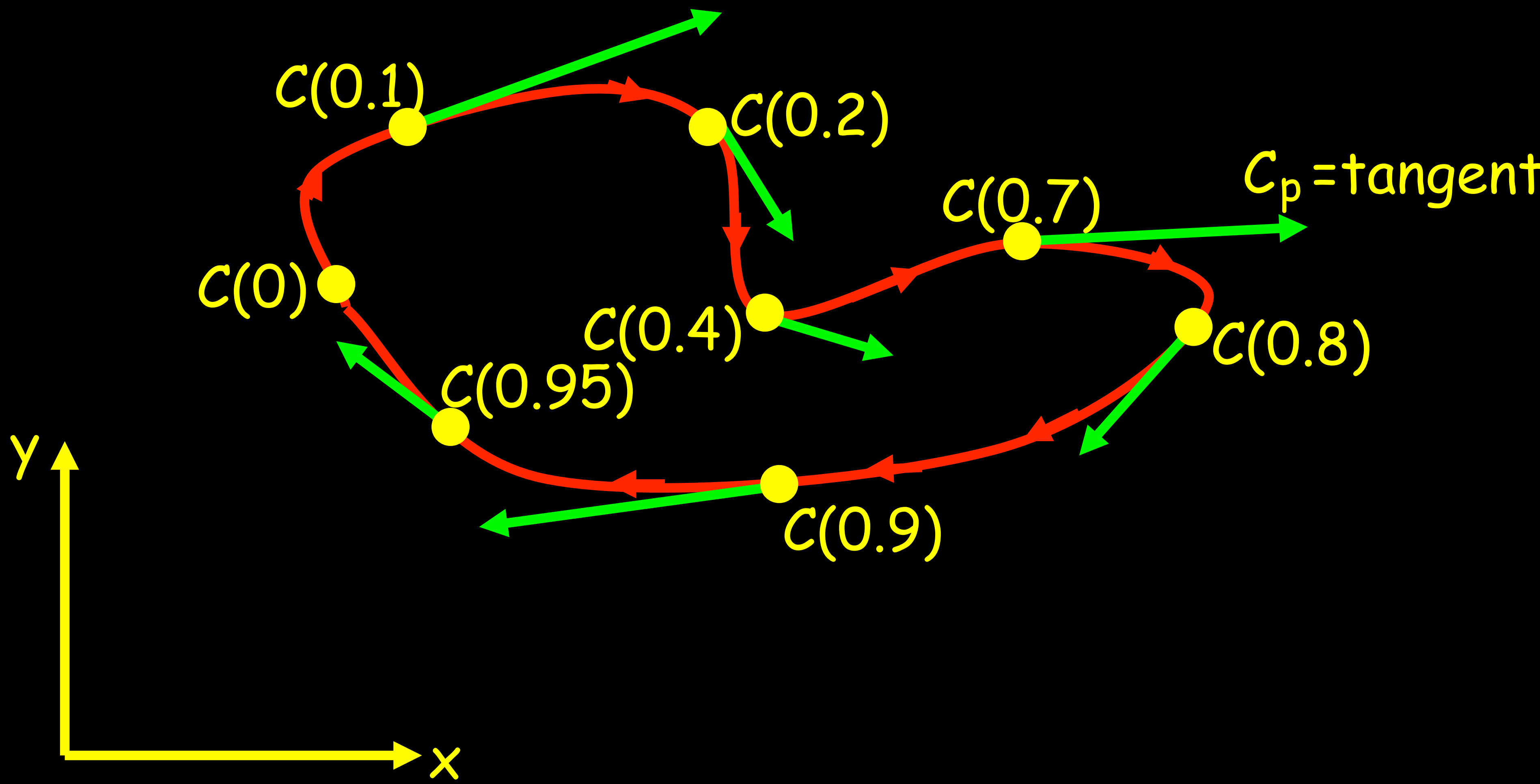


$$\min_{\{\psi_j\}} \sum_{i=1}^{N} \left| F_i - \sum_{j=1}^{k} \langle F_i, \psi_j \rangle \psi_j \right|^2$$

Aflalo, Brezis, Bruckstein, K., Sochen

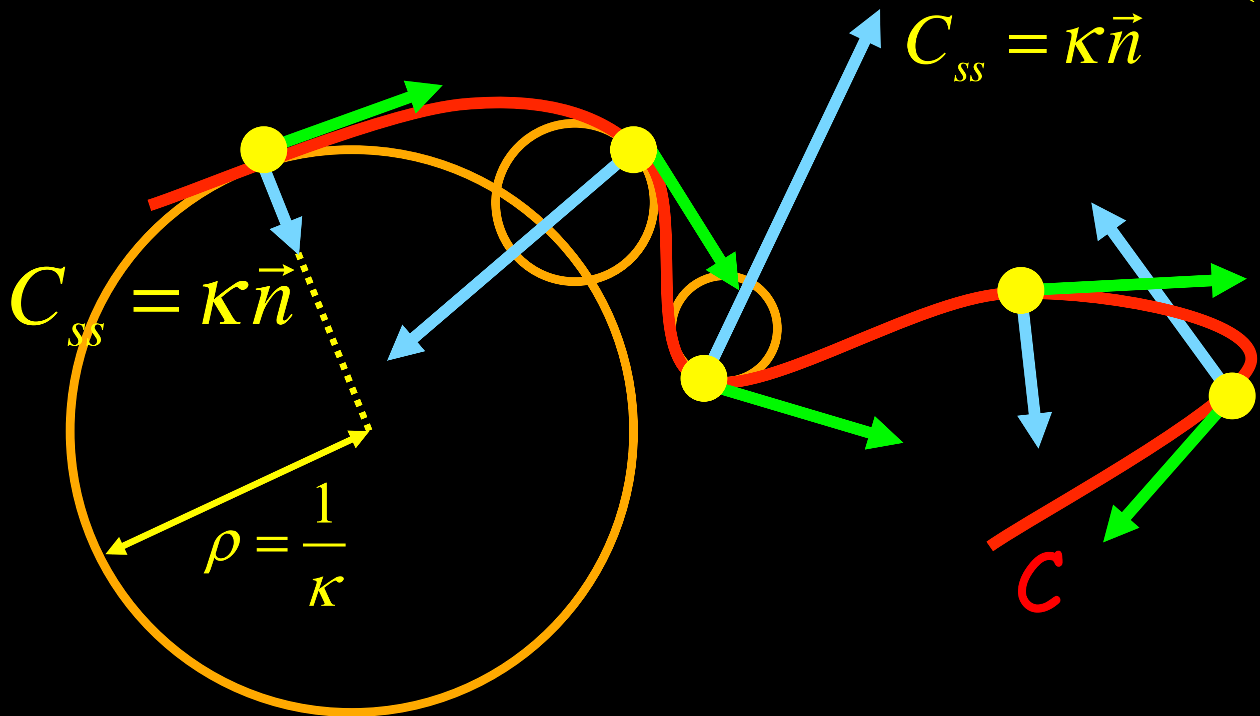
Planar Curves

- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$



Arc-length and Curvature

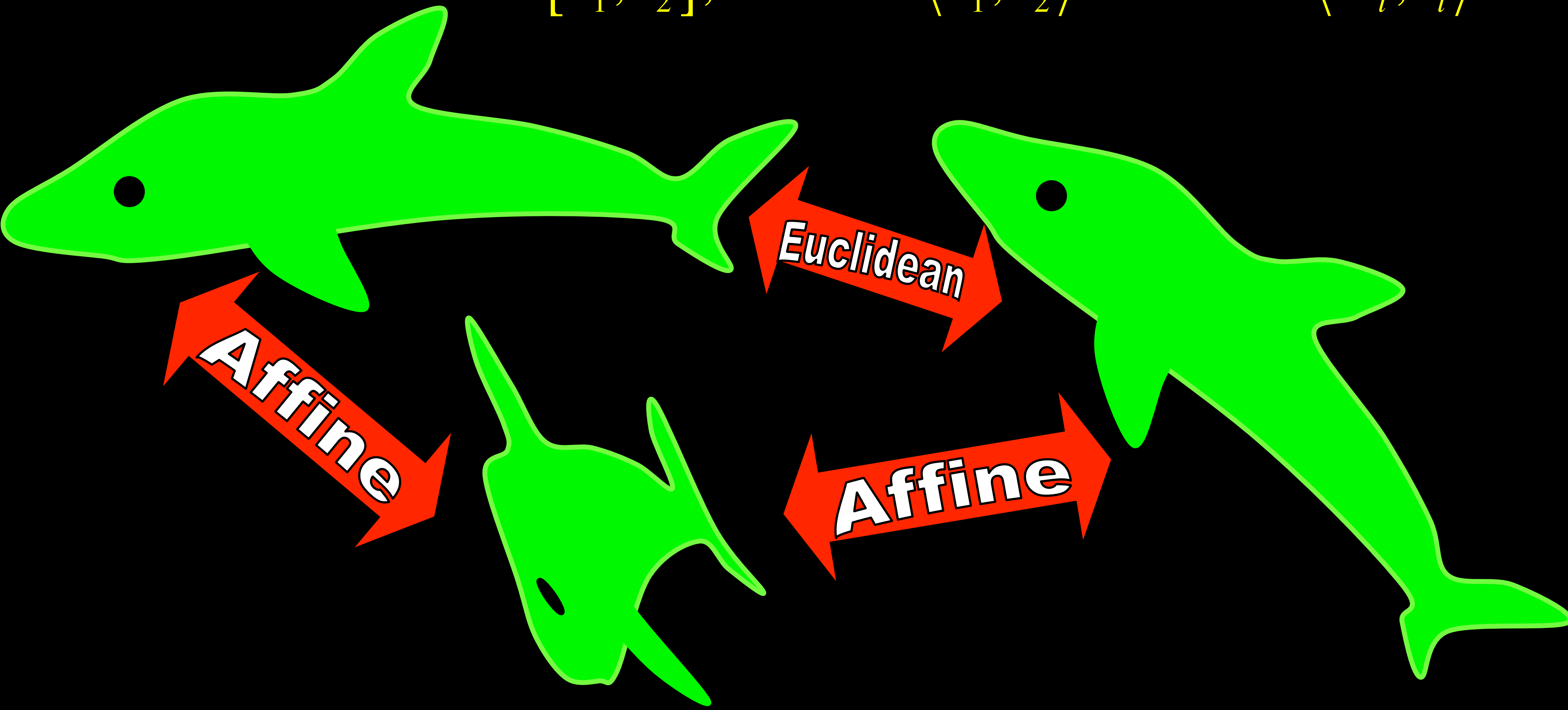
$$s(p)=\int_0^p |C_p|dp \Longrightarrow |C_s|=1, \quad \left(\vec{t}=C_s=\frac{C_p}{|C_p|}\right)$$



Linear Transformations

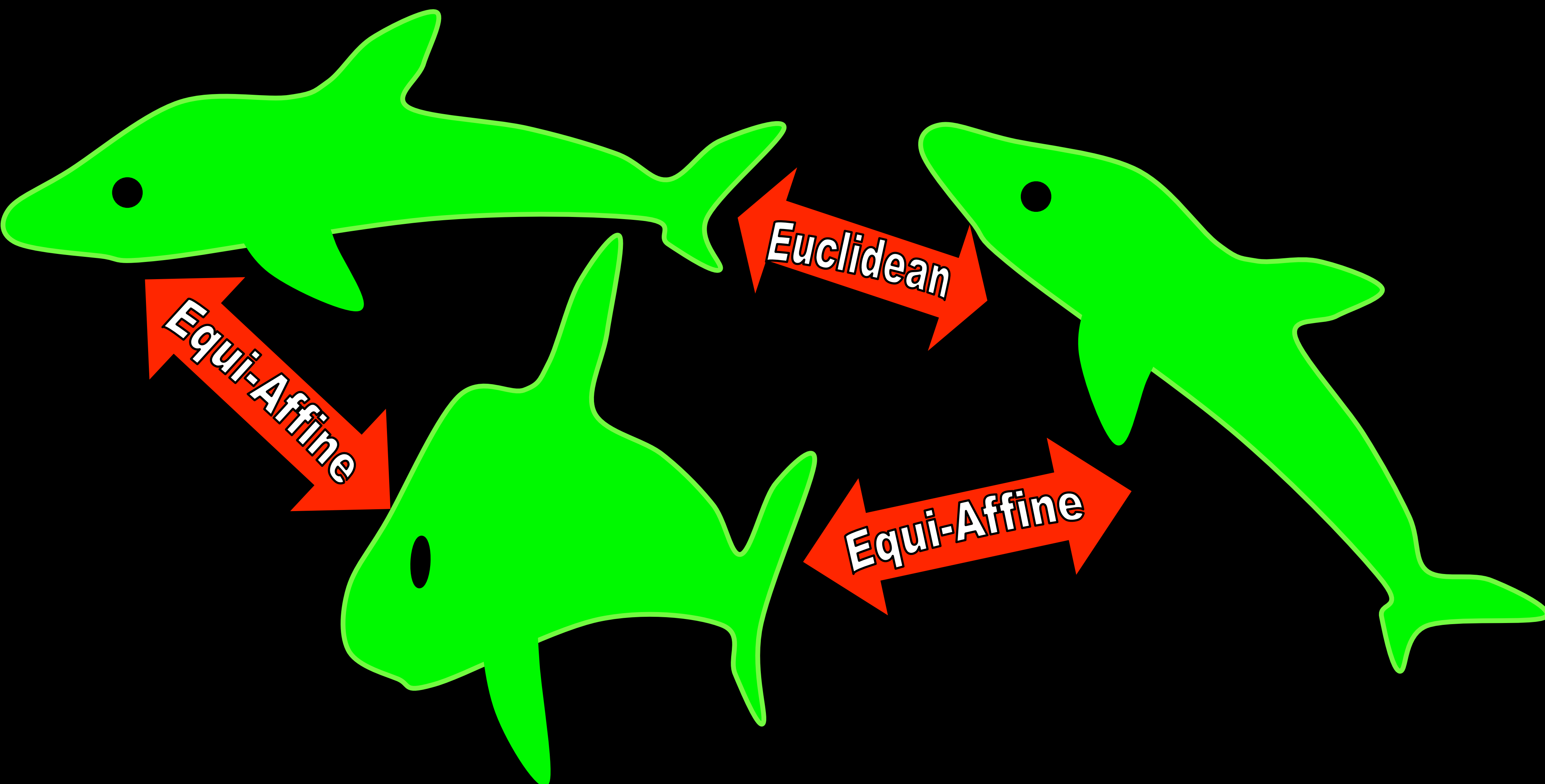
Affine: $\{\tilde{x},\tilde{y}\}^T = A\{x,y\}^T + \bar{b},$

Euclidean: $A=[\bar{u}_1,\bar{u}_2]$, where $\langle \bar{u}_1,\bar{u}_2 \rangle = 0$ and $\langle \bar{u}_i,\bar{u}_i \rangle = 1.$



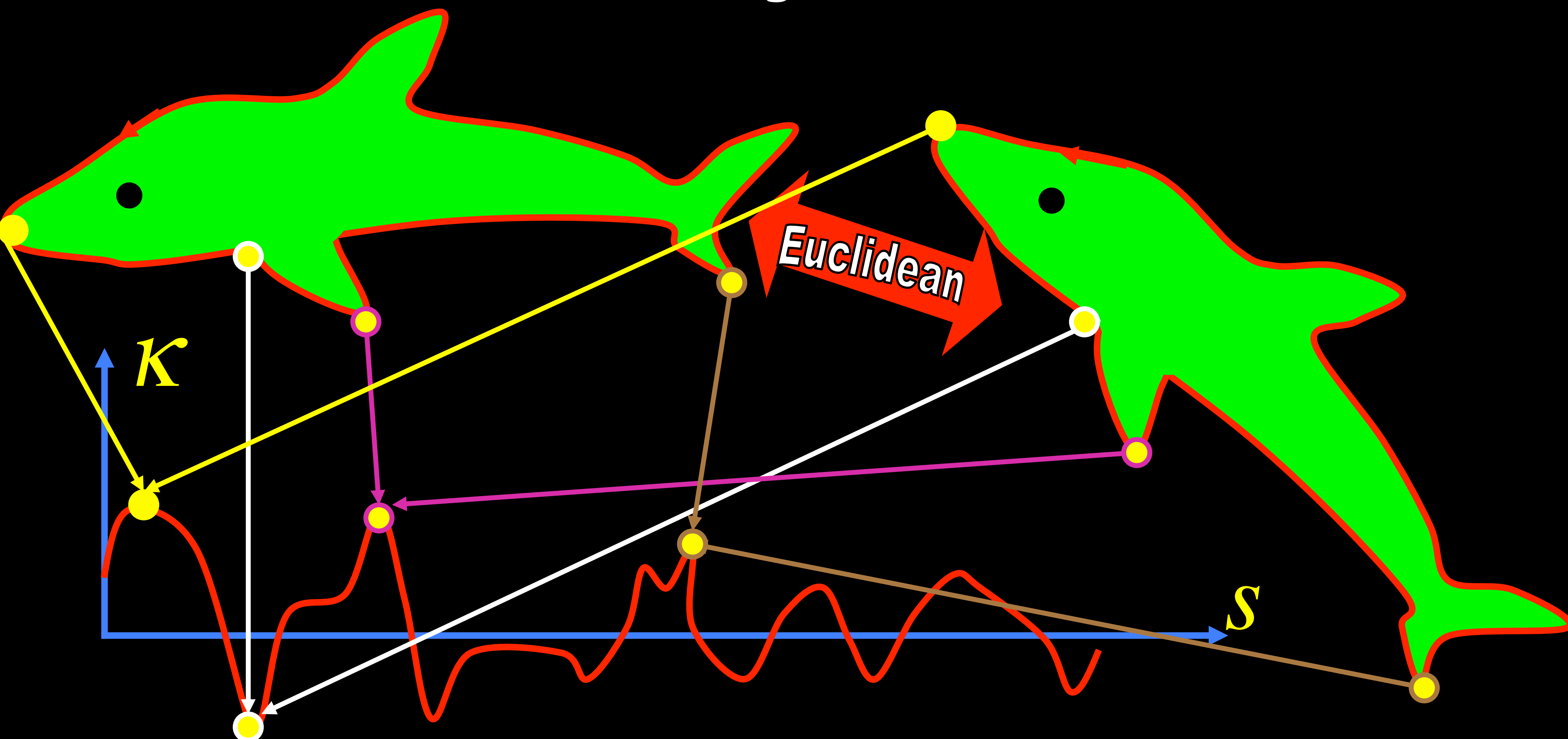
Linear Transformations

Equi-Affine: $\{\tilde{x},\tilde{y}\}^T = A\{x,y\}^T + \bar{b}, \quad \det(A) = 1.$



Differential Signatures

- Euclidean invariant signature $\{s, \kappa(s)\}$



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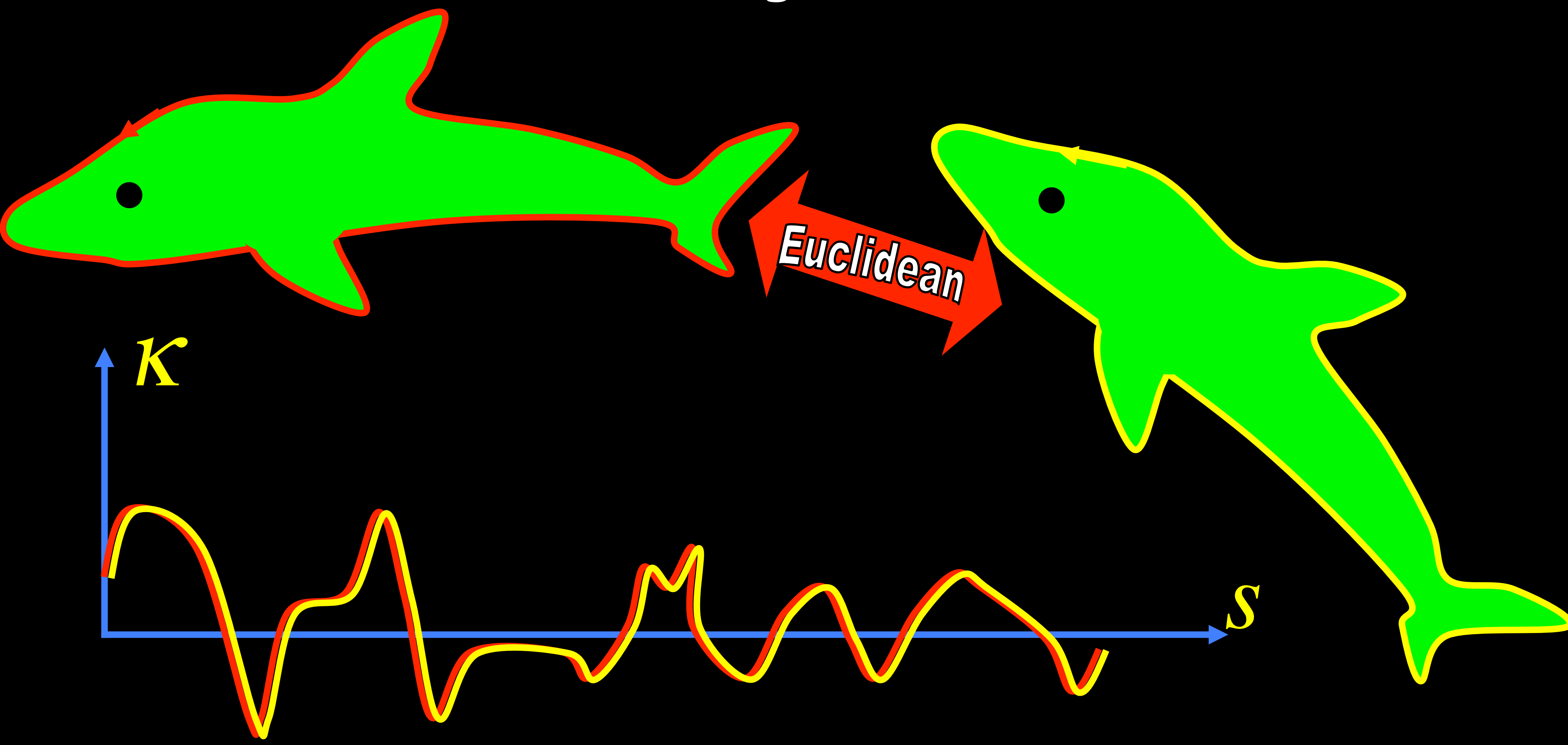
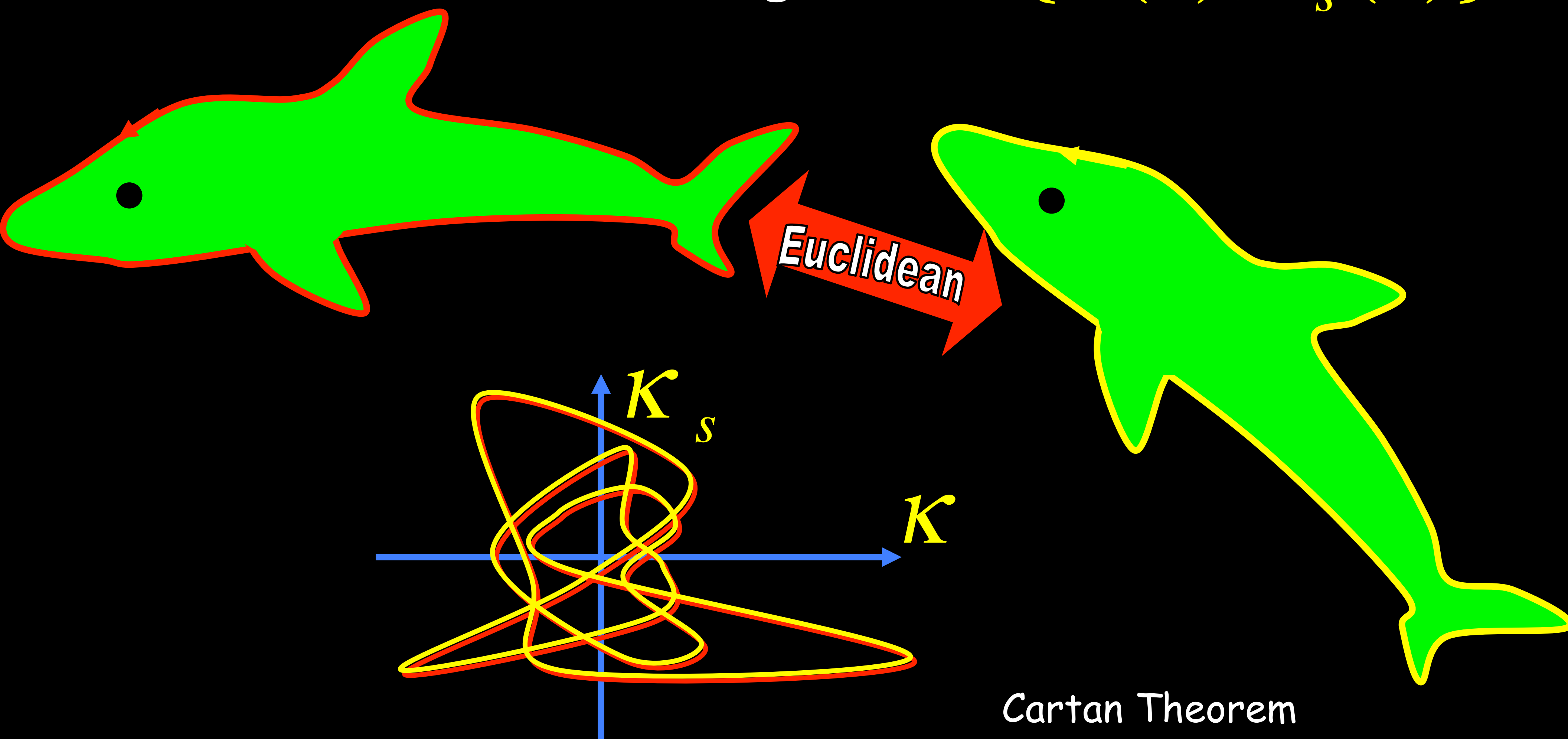


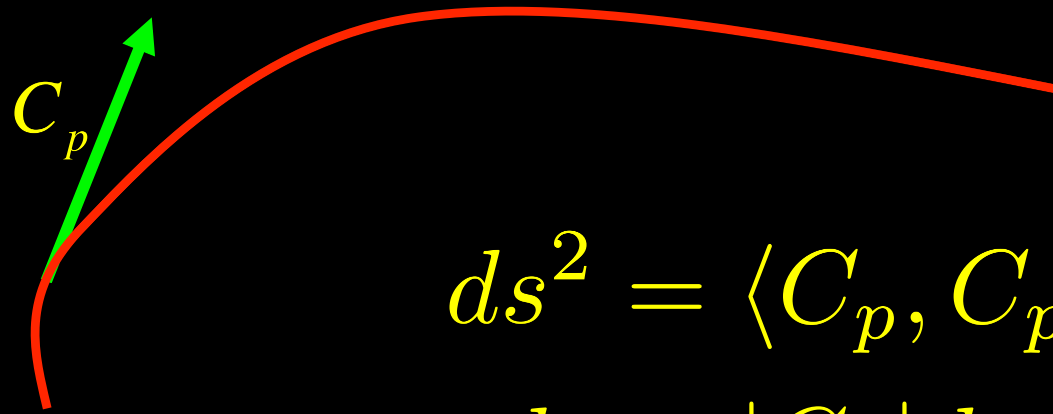
Differential Signatures

- Euclidean invariant signature $\{\kappa(s), \kappa_s(s)\}$



Euclidean arclength

- Length is preserved, thus $$1 = \langle C_s, C_s \rangle = \langle C_p p_s, C_p p_s \rangle = \langle C_p, C_p \rangle p_s^2$$



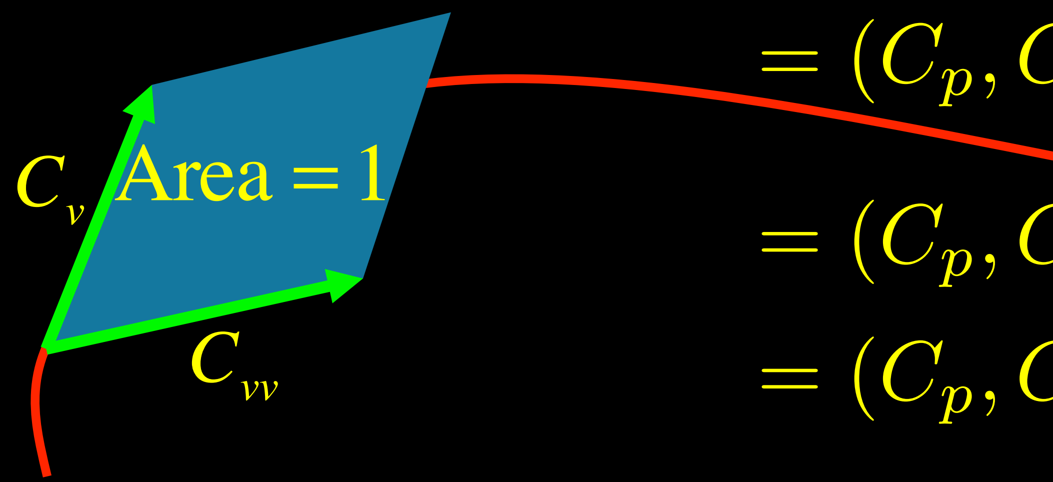
$$ds^2 = \langle C_p, C_p \rangle dp^2$$

$$ds = |C_p| dp$$

Equi-affine arclength

- Area is preserved,

$$
\begin{aligned}
1 &= (C_v, C_{vv}) \\
&= (C_p p_v, \frac{d}{dp}(C_p p_v) p_v) \\
&= (C_p, C_{pp} p_v + C_p \frac{d}{dp} p_v) p_v^2 \\
&= (C_p, C_{pp} p_v) p_v^2 \\
&= (C_p, C_{pp}) p_v^3
\end{aligned}
$$

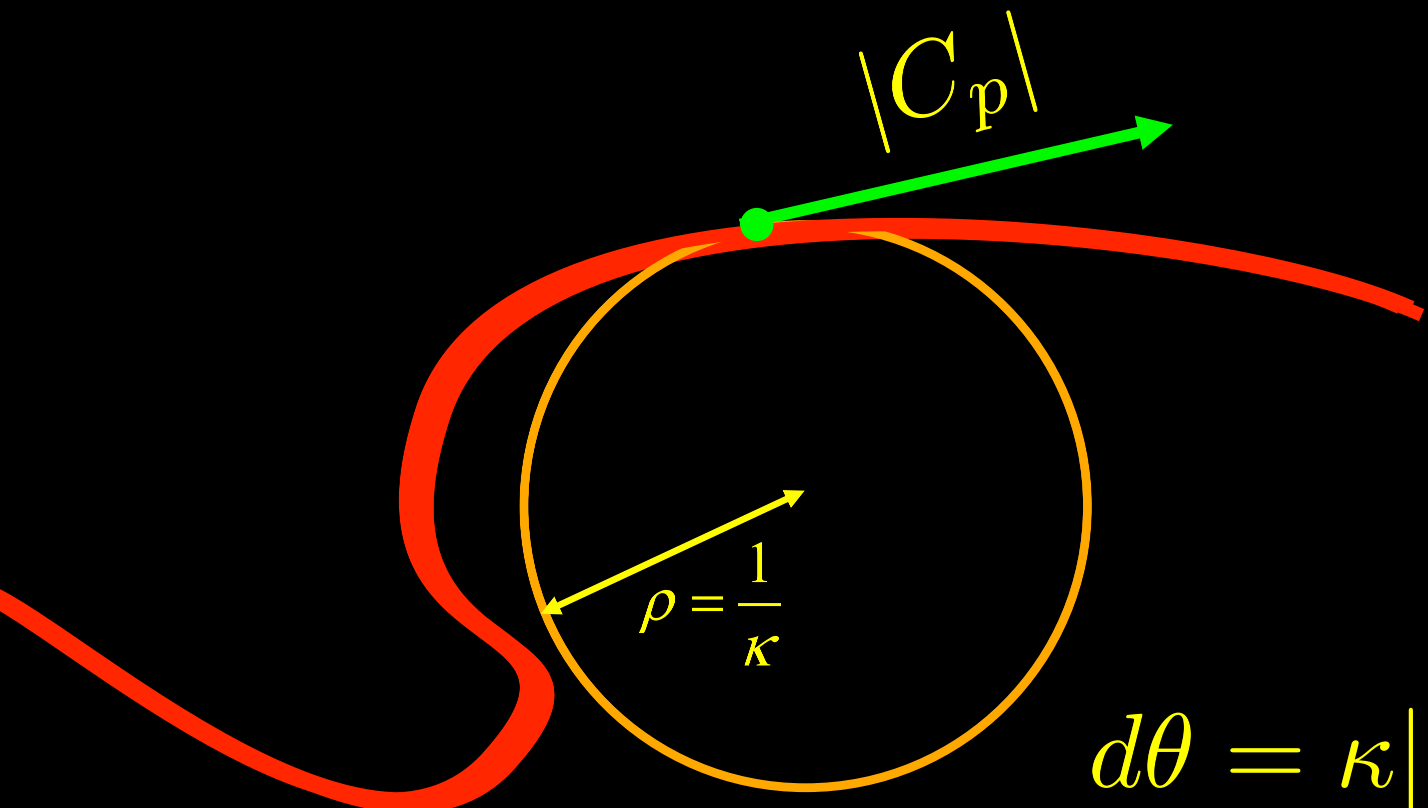


$$dv = (C_p, C_{pp})^{1/3} dp = (C_s, C_{ss})^{1/3} ds = |\kappa|^{1/3} ds$$

Scale inv. arclength

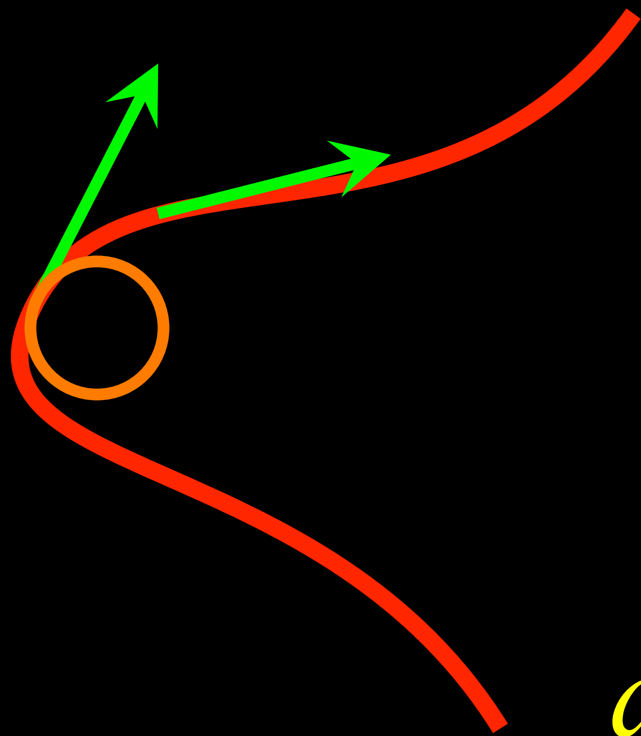
- Ratio is preserved,

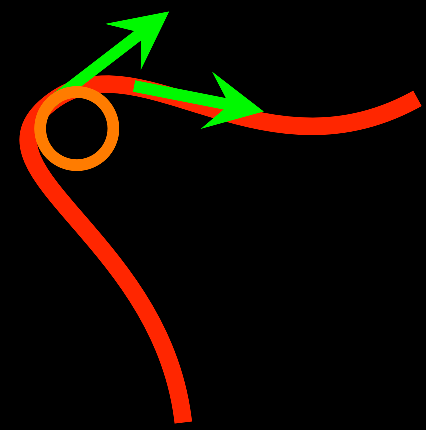
$$1 = \frac{|C_\theta|}{\rho}$$
$$= \kappa |C_\theta|$$
$$= \kappa |C_p p_\theta|$$
$$= \kappa |C_p| p_\theta$$



$$d\theta = \kappa |C_p| dp = \kappa ds$$

Scale invariance?





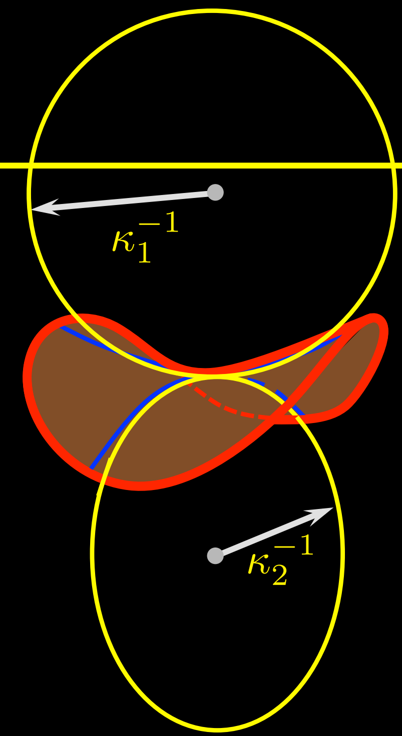
$$d\theta = |\kappa| ds$$

Interesting transformations

Lewis Carroll, Alice $\in$ Wonderland

Disney, Alice $\in$ Wonderland

From curves to surfaces



$$K = \kappa_1 \kappa_2$$

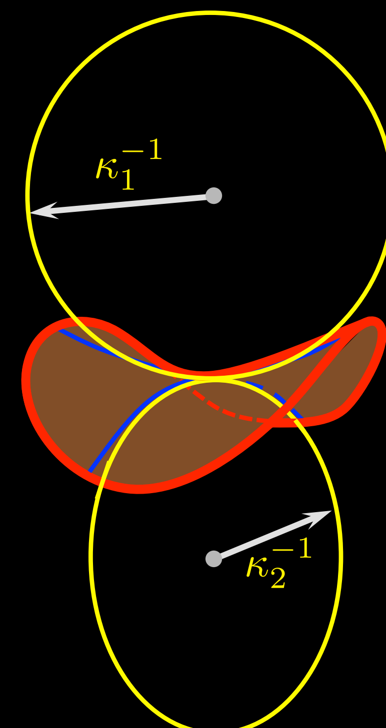
$$S(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$dx^2 = (x_u du + x_v dv) = x_u^2 du^2 + 2 x_u x_v du dv + x_v^2 dv^2$$

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (du\, dv) \begin{pmatrix} S_u^2 & \langle S_u, S_v \rangle \\ \langle S_u, S_v \rangle & S_v^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \\ &= (du\, dv) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \end{aligned}$$

From curves to surfaces

$$
\begin{aligned}
d\tilde{s}^2 &= \kappa_1\kappa_2(dx^2+dy^2+dz^2)\\
&= (du\,dv)\begin{pmatrix} K S_u^2 & K\langle S_u, S_v\rangle \\ K\langle S_u, S_v\rangle & K S_v^2 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}\\
&= (du\,dv)\,(K g_{ij})\begin{pmatrix} du \\ dv \end{pmatrix}
\end{aligned}
$$

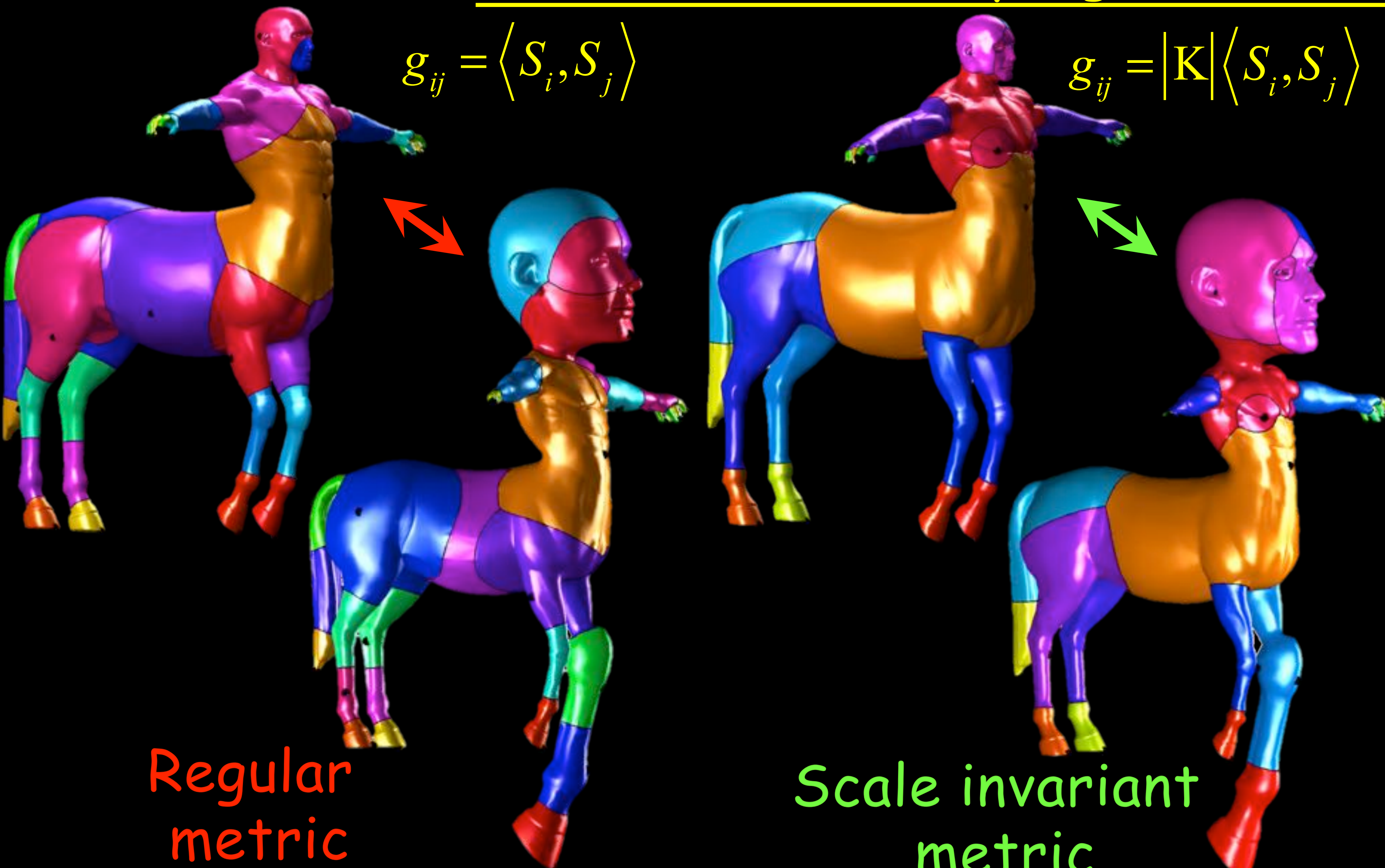


$$\tilde{g}_{ij} = |K| g_{ij} = |\kappa_1\kappa_2|\langle S_i, S_j\rangle$$

Farthest Point Sampling - Voronoi

$$g_{ij} = \langle S_i, S_j \rangle$$

$$g_{ij} = |K| \langle S_i, S_j \rangle$$



Regular metric

Scale invariant metric

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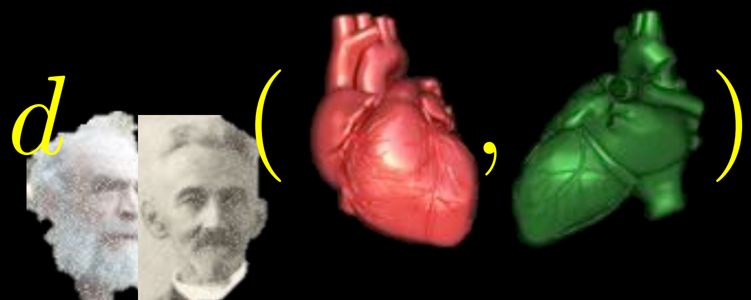
Generalized MDS

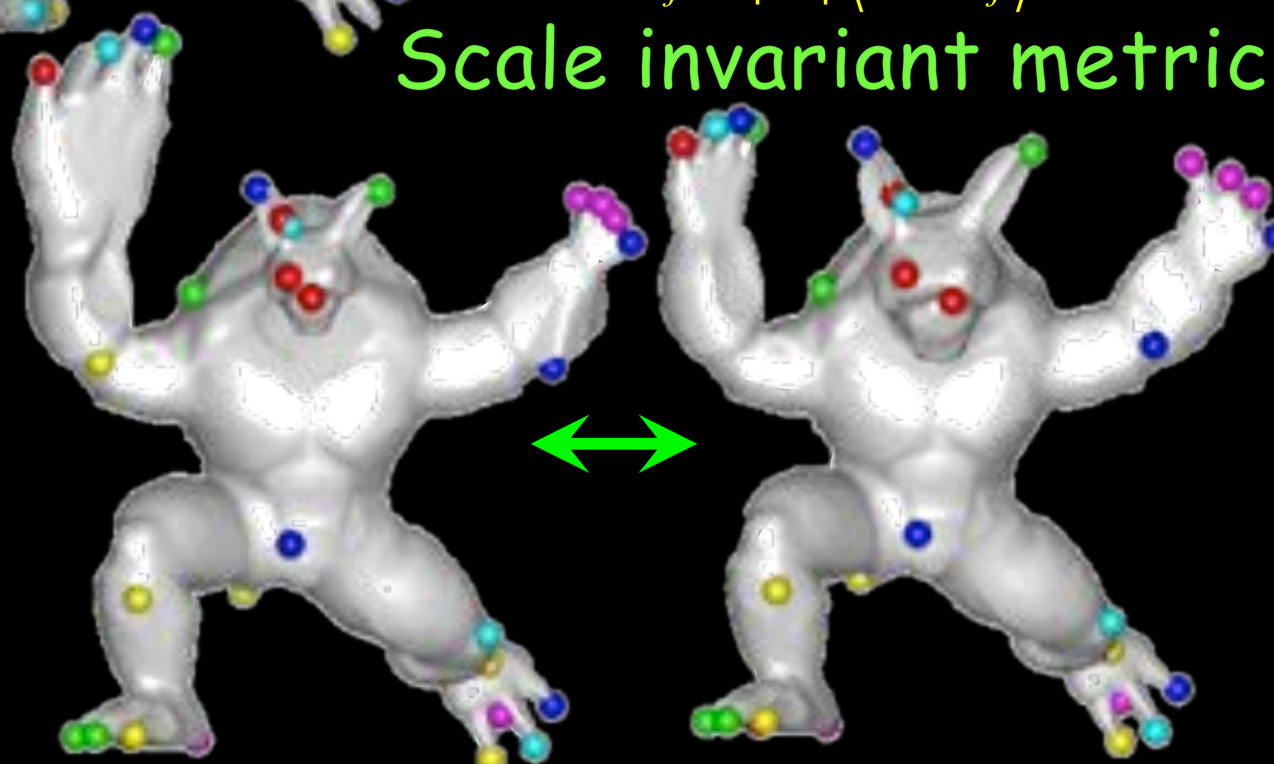
$$g_{ij} = |\mathrm{K}| \langle S_i, S_j \rangle$$

Scale invariant metric

Regular metric

$$g_{ij} = \langle S_i, S_j \rangle$$





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Circulant Matrix Decomposition

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

Circulant Matrix Decomposition

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix}$$

$$C = c_0 I + c_1 P + c_2 P^2 + \ldots + c_{n-1} P^{n-1}$$

$$P = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

Circulant Matrix Decomposition

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix}.$$

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$$P u_j = \lambda_j u_j$$

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix}.$$

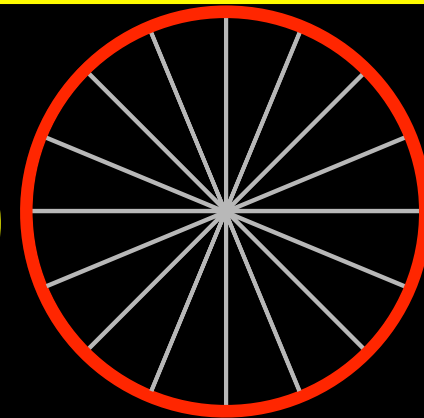
$$Pu_j = \lambda_j u_j$$

Circulant Matrix Decomposition

$$C = c_0 I + c_1 P + c_2 P^2 + \ldots + c_{n-1} P^{n-1}$$

$$P = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

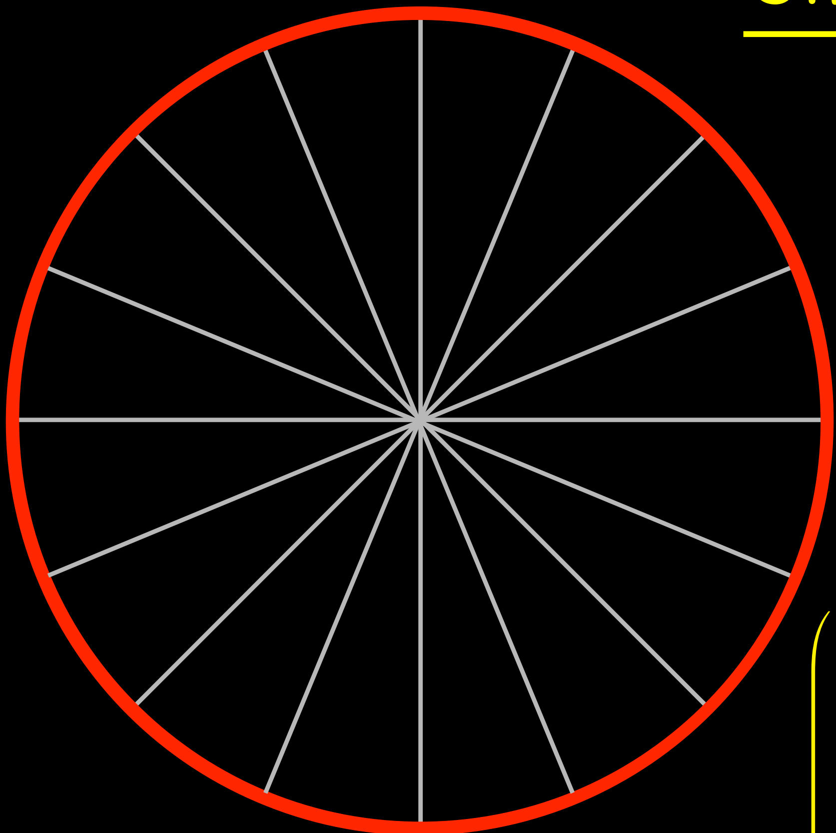
$$\omega_j = \exp\left(\frac{2\pi i j}{n}\right)$$



$$u_j = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \omega_j \\ \omega_j^2 \\ \vdots \\ \omega_j^{n-1} \end{bmatrix}, \quad j = 0, 1, \ldots, n-1,$$

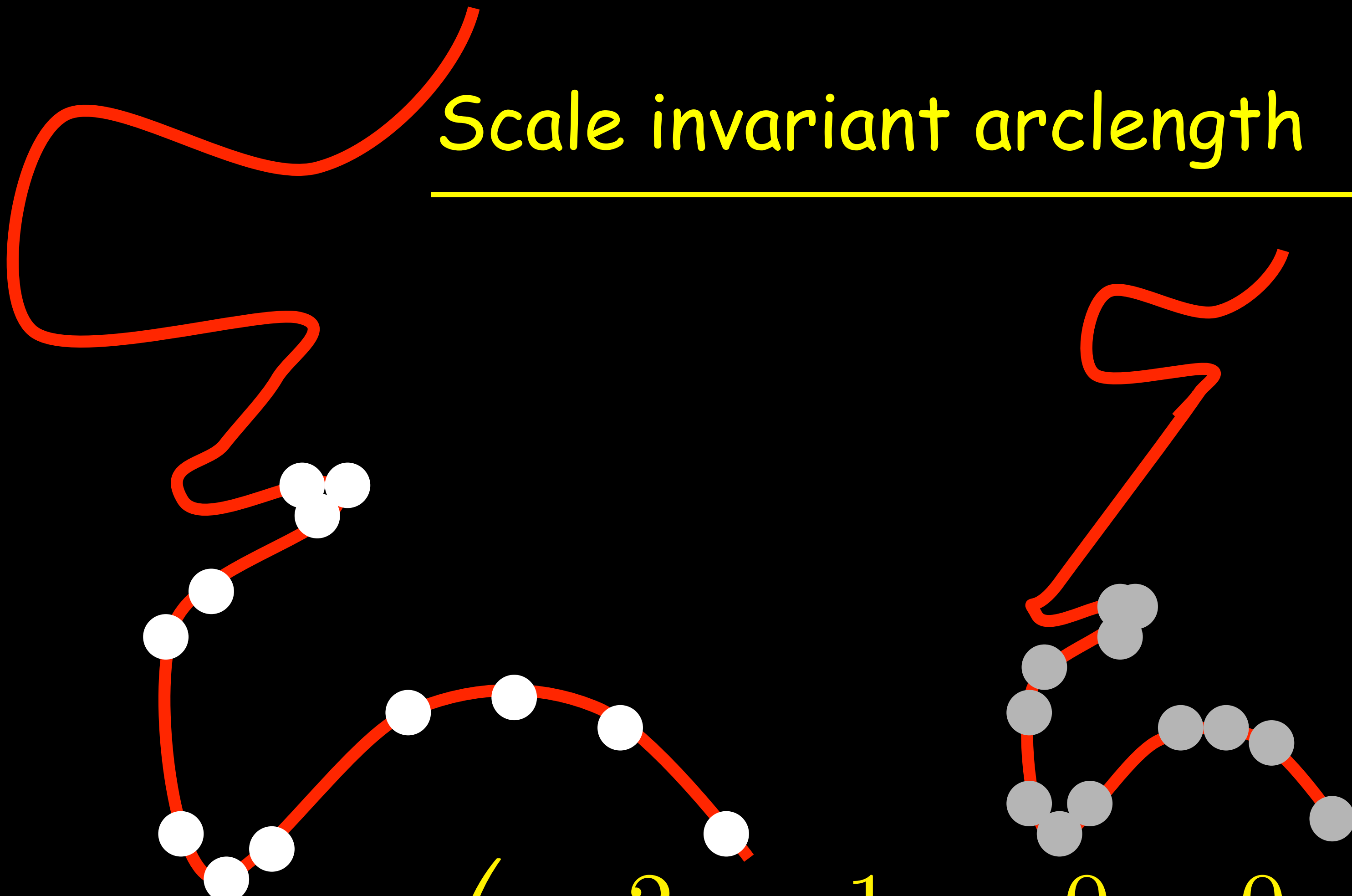
$$\lambda_j = c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \ldots + c_1 \omega_j^{n-1}, \qquad j = 0, 1, \ldots, n-1$$

On the origin of Fourier transform



$$\frac{d^2}{d\theta^2} \approx \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}$$

Scale invariant arclength



$$\frac{d^2}{d\theta^2} \approx \begin{pmatrix} -2 & 1 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & \ldots \\ 0 & 1 & -2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

From curves to surfaces

$$\Delta_{\tilde g} \equiv -\frac{1}{\sqrt{\tilde g}}\partial_i \sqrt{\tilde g}\,\tilde g^{ij}\partial_j$$

$\tilde g_{ij} =$  $= |\kappa_1 \kappa_2| \langle S_i, S_j \rangle$

Regular metric $g_{ij} = \langle S_i, S_j \rangle$

$\Delta_g \phi_i = \lambda_i \phi_i$ Eigenfunctions

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Scale invariant metric $g_{ij} = |K| \langle S_i, S_j \rangle$

$\Delta_g \phi_i = \lambda_i \phi_i$ Eigenfunctions

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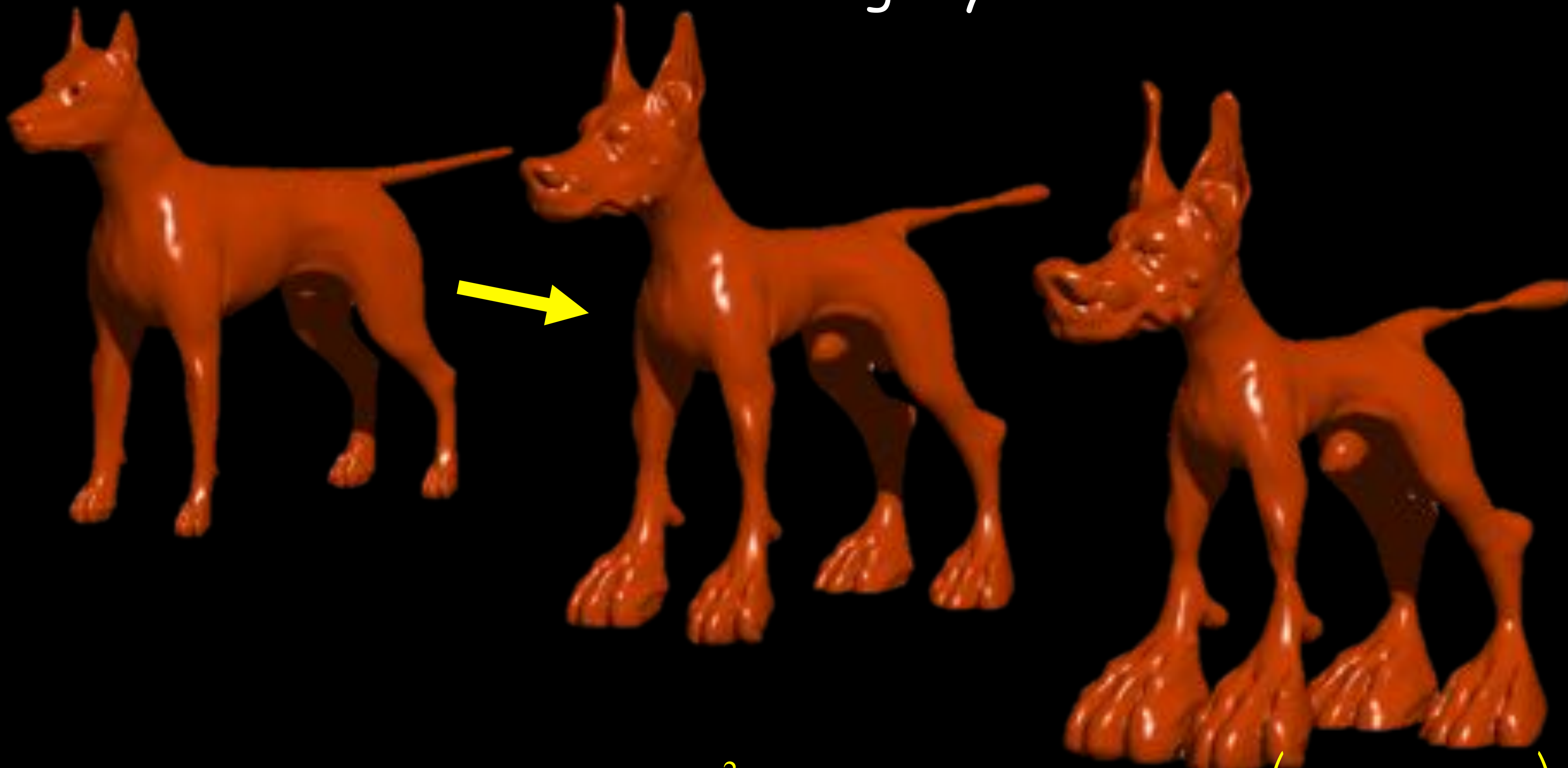
Self caricaturization

- Coordinates scaling by the Gaussian curvature



$$\int_S \left\| \nabla_G \tilde{S} - |\mathrm{K}|^\alpha \nabla_G S \right\|^2 da$$

$$\Delta_G \tilde{S} = \nabla_G \cdot \left(|\mathrm{K}|^\alpha \nabla_G S \right)$$

Sela, Aflalo, K. CVIU

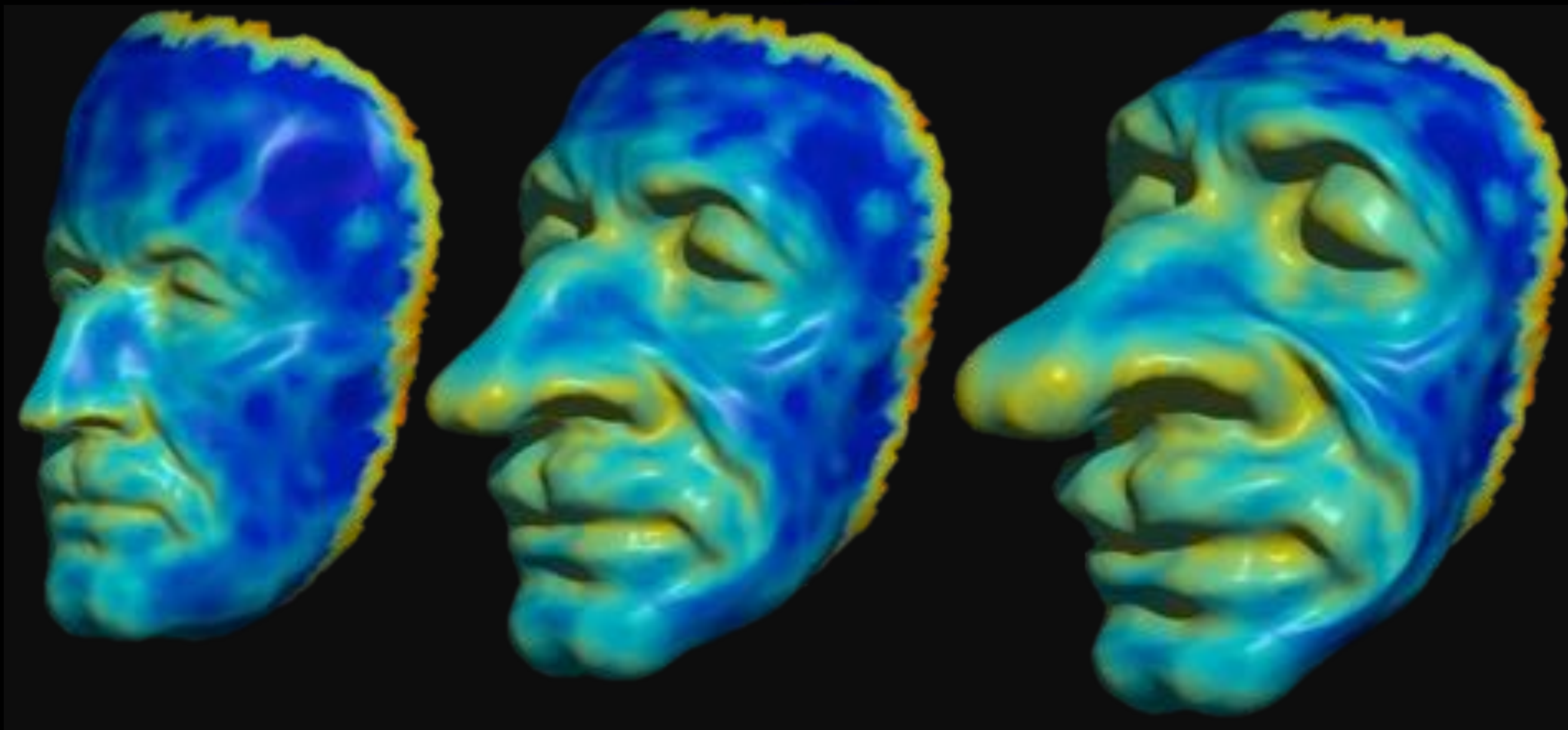
Self caricaturization

- Scaling by the Gaussian curvature

$$\int_S \left\| \nabla_G \tilde{S} - |\mathrm{K}|^{\alpha} \nabla_G S \right\|^2 da \qquad \Delta_G \tilde{S} = \nabla_G \cdot \left(|\mathrm{K}|^{\alpha} \nabla_G S \right)$$

Sela, Aflalo, K. CVIU

Self caricaturization



$$\int_S \left\| \nabla_G \tilde{S} - |\mathrm{K}|^{\alpha} \nabla_G S \right\|^2 da \qquad \Delta_G \tilde{S} = \nabla_G \cdot \left(|\mathrm{K}|^{\alpha} \nabla_G S \right)$$

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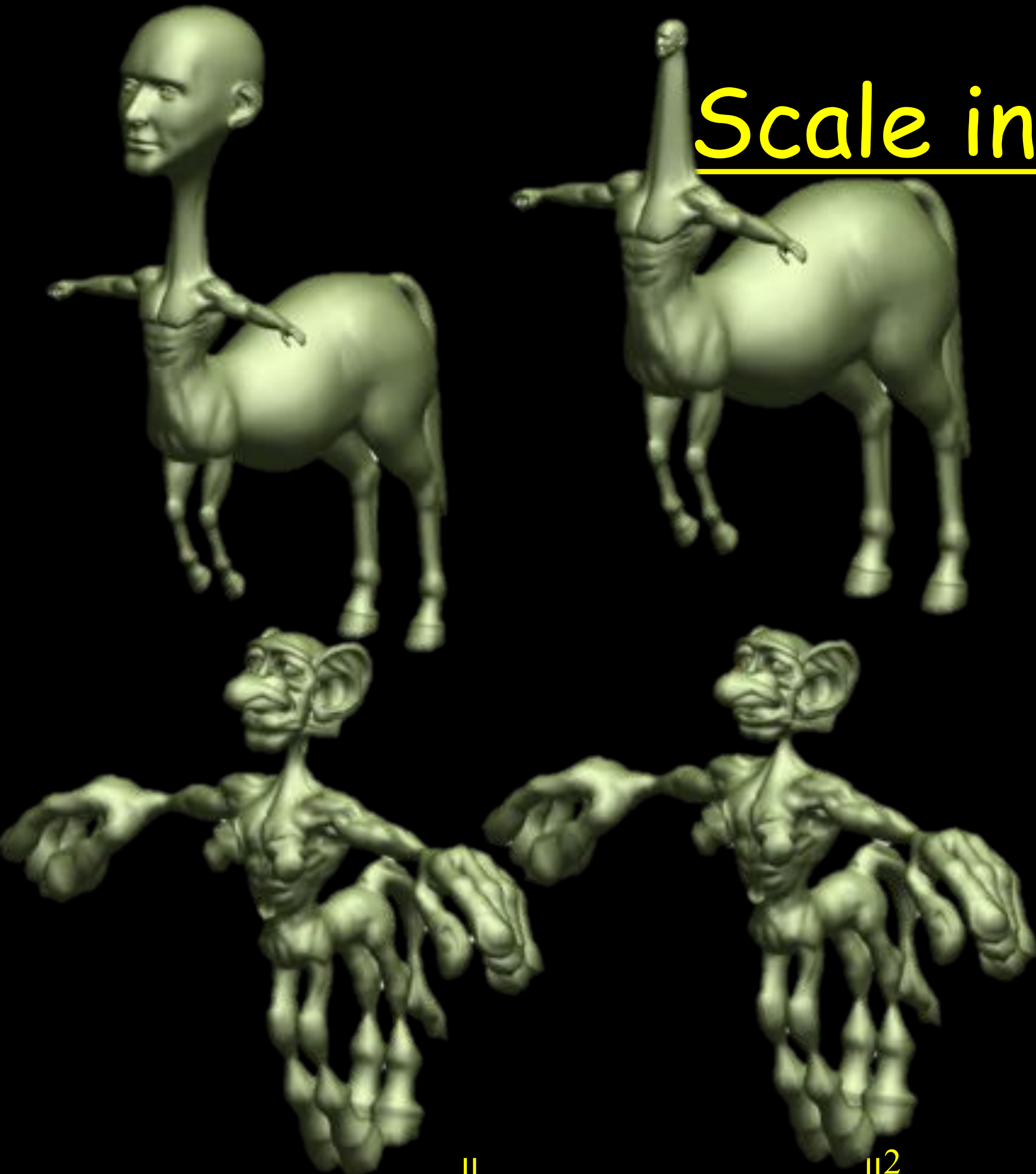
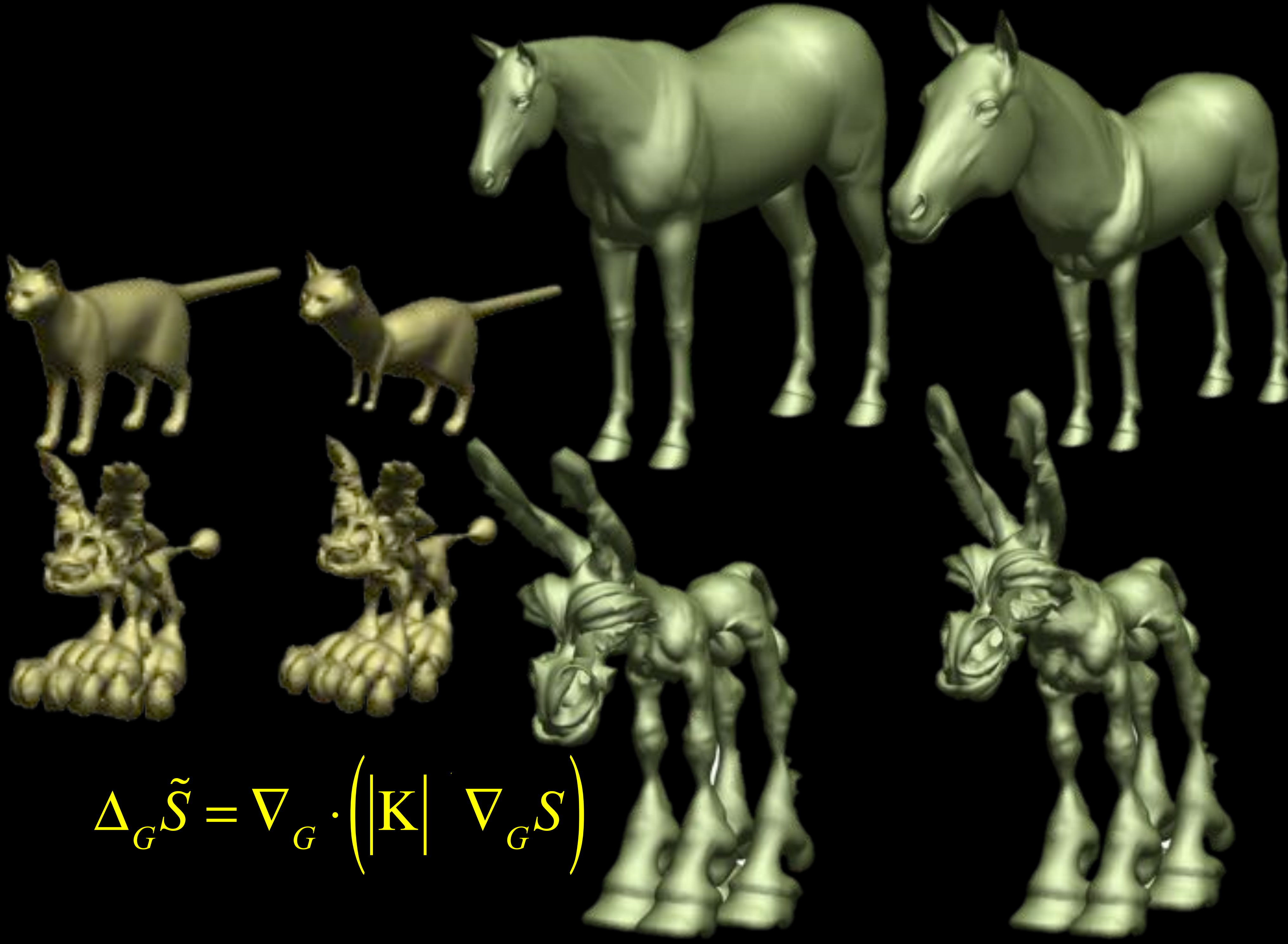
Scale inv. canonical forms



$$\int_S \left\| \nabla_G \tilde{S} - |\mathrm{K}| \;\; \nabla_G S \right\|^2 da$$

$$\Delta_G \tilde{S} = \nabla_G \cdot \left(|\mathrm{K}| \;\; \nabla_G S \right)$$

Sela, Aflalo, K. CVIU



$$\Delta_G \tilde{S} = \nabla_G \cdot \left(\left| \mathrm{K} \right| \; \nabla_G S \right)$$

Sela, Aflalo, K. CVIU

Δ_g

Optimality of the spectral domain

The LBO spectral domain is optimal in approximating functions with bounded gradient magnitude.

Let S be a given Riemannian manifold with a metric (g_{ij}), the induced LBO, Δ_g, with associated spectral basis $\{\phi_i\}$, where

$$\Delta_g \phi_i = \lambda_i \phi_i$$

For any $f : S \to \mathbb{R}$, the representation error

$$\|r_n\|_g^2 \equiv \|f - \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i\|_2^2 \leq \frac{\|\nabla_g f\|_2^2}{\lambda_{n+1}}$$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$$

Aflalo, K. [PNAS]

Optimality of the spectral domain

By the Courant-Fischer min-max principle, there is no scalar

$0 \le \alpha < 1$ and a basis $\{\psi_i\}_{i=1}^{\infty}$, such that for any f

$$\min_{\{\psi_i\}_0^{\infty}} \left\| f - \sum_{i=1}^{n} \langle f, \psi_i \rangle \psi_i \right\|_2^2 \le \alpha \frac{\|\nabla_g f\|_2^2}{\lambda_{n+1}}$$

The natural spectral domain is optimal for approximating smooth functions.

Aflalo, Brezis, K. SIAM-IS

Shape representation

- Original

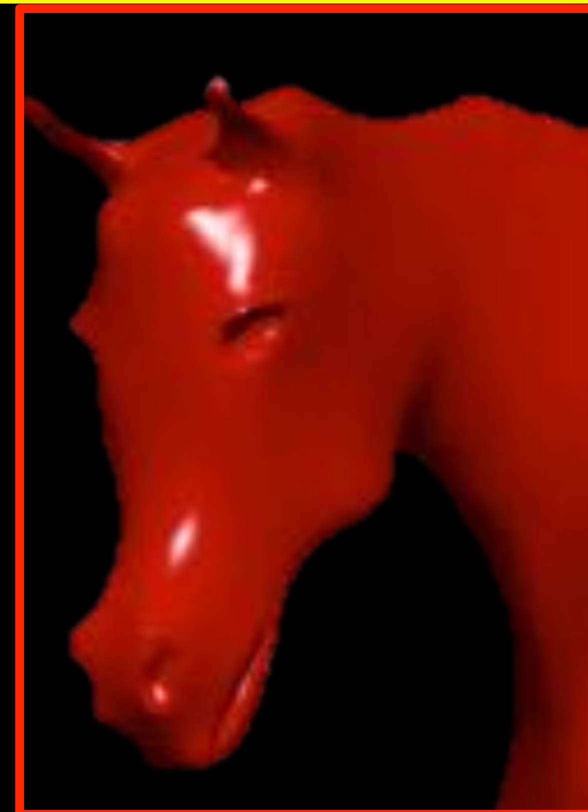
- Reconstruction

$$g_{ij} = \langle S_i, S_j \rangle \quad \tilde{\tilde{g}}_{ij} = |K|^{0.4} g_{ij} \quad \tilde{g}_{ij} = |K| g_{ij}$$

$$\Delta_g \phi_i = \lambda_i \phi_i \quad \hat{S} \approx \sum_{i=1}^{300} \langle S, \phi_i \rangle \phi_i$$

Aflalo, Brezis, K. . SIAM-IS

Shape representation



$$g_{ij} = \langle S_i, S_j \rangle \qquad \tilde{\tilde{g}}_{ij} = |K|^{0.4} g_{ij} \qquad \tilde{g}_{ij} = |K| g_{ij}$$

$$\Delta_g \phi_i = \lambda_i \phi_i \qquad \hat{S} \approx \sum_{i=1}^{300} \langle S, \phi_i \rangle \phi_i$$

Aflalo, Brezis, K. . SIAM-IS

Principal Component Analysis

- Given $\{x_i\}_{i=1}^{k}$
- Find $\mathbf{P}$ optimized for $\min_{\mathbf{P}} \sum_{i=1}^{k} \left\| \mathbf{P}\mathbf{P}^T x_i - x_i \right\|_2^2$

$$\text{s.t. } \mathbf{P}^T\mathbf{P} = \mathbf{I}_m$$

- LBO eigenfunctions optimize the Dirichlet energy

$$\mathbf{\Phi} = \arg \min_{\{\phi_i\}_1^n} \sum_{i=1}^{n} \|\nabla_g \phi_i\|_g^2$$

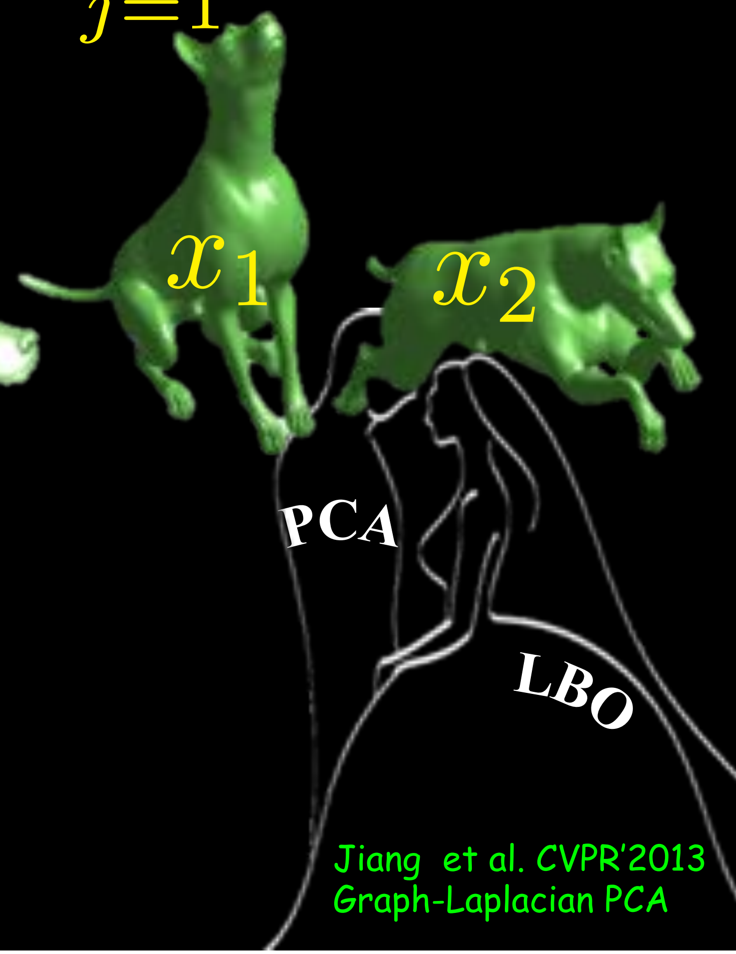
$$\text{s.t. } \langle \phi_i, \phi_j \rangle_g = \delta_{ij}, \ \forall (i,j)$$

Regularized PCA (by LBO)

PCA LBO

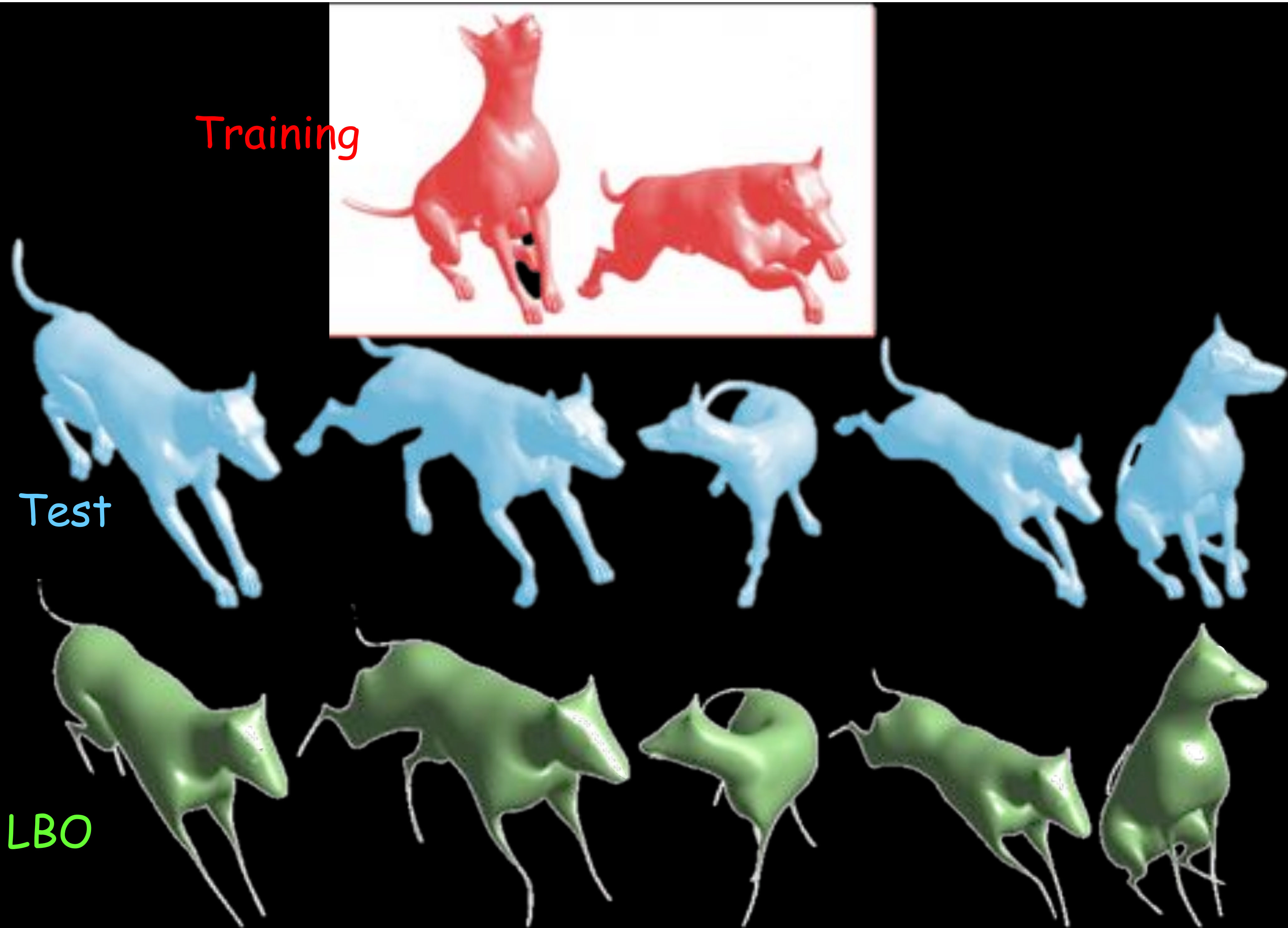
$$\min_{\mathbf{P}} \sum_{i=1}^{k} \left\| \mathbf{P}\mathbf{P}^T \mathbf{A} x_i - x_i \right\|_g^2 + \mu \sum_{j=1}^{m} \| \nabla_g P_j \|_g^2$$

$$\text{s.t.} \ \ \mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{I}_m$$



Aflalo, Brezis, K. . SIAM-IS 2015

Jiang et al. CVPR'2013
Graph-Laplacian PCA



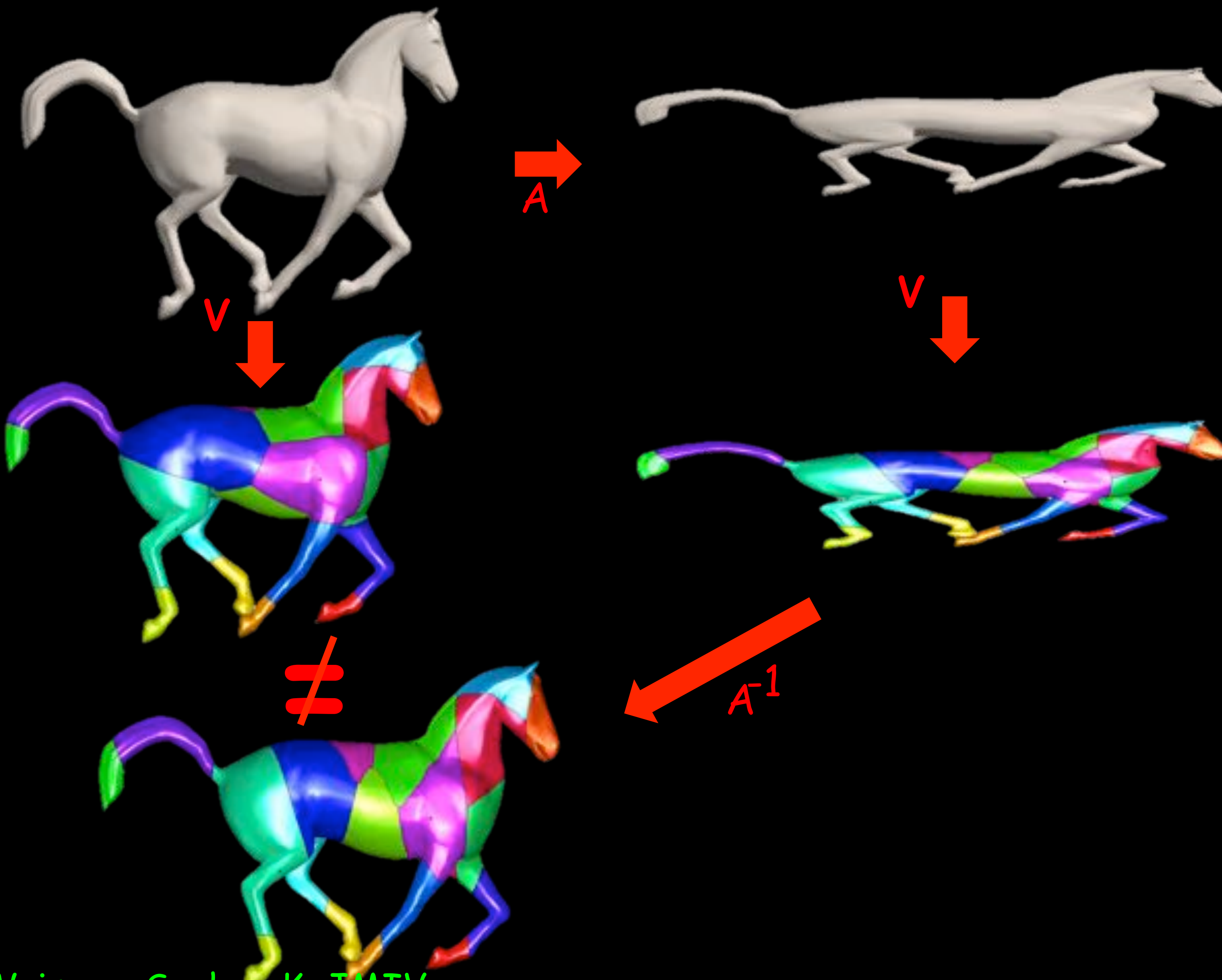
Aflalo, Brezis, K. SIAM-IS


Training
Test
PCA
Reg.
PCA
Aflalo, Brezis, K. SIAM-IS

$$\left|\nabla_g d\right| = 1$$

$$g_{ij} = \left\langle S_i, S_j \right\rangle$$

Voronoi Diagrams



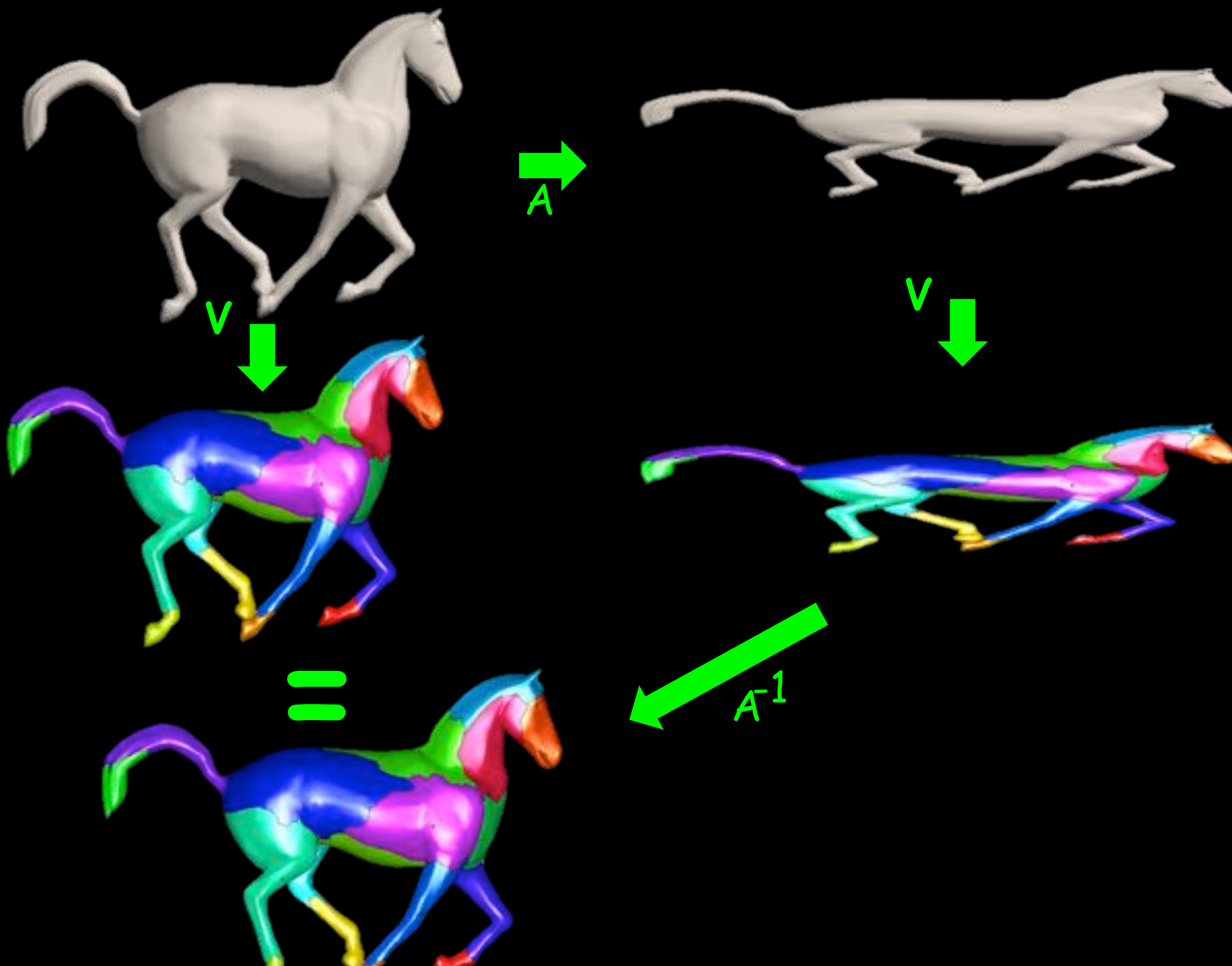
Raviv Bronstein^2 Waisman Sochen K. JMIV

$$\left|\nabla_g d\right| = 1$$

Equi-Affine invariant Voronoi Diagrams

$$\tilde{g}_{ij} = \det\left(S_u, S_v, S_{ij}\right)$$

$$g_{ij} = \tilde{g}_{ij}\tilde{g}^{-1/4}$$



Raviv Bronstein[2] Waisman Sochen K. JMIV

Axiomatic invariants - curves

Euclidean $ds = |C_p| dp$ $g = \langle C_p, C_p \rangle^{1/2}$

Equi-affine $dv = (C_p . C_{pp})^{1/3} dp$ $g = (C_p . C_{pp})^{1/3}$

Scale $d\theta = \kappa |C_p| dp$ $g = \kappa |C_p|$

Euclidean $g_{ij} = \langle S_i, S_j \rangle$

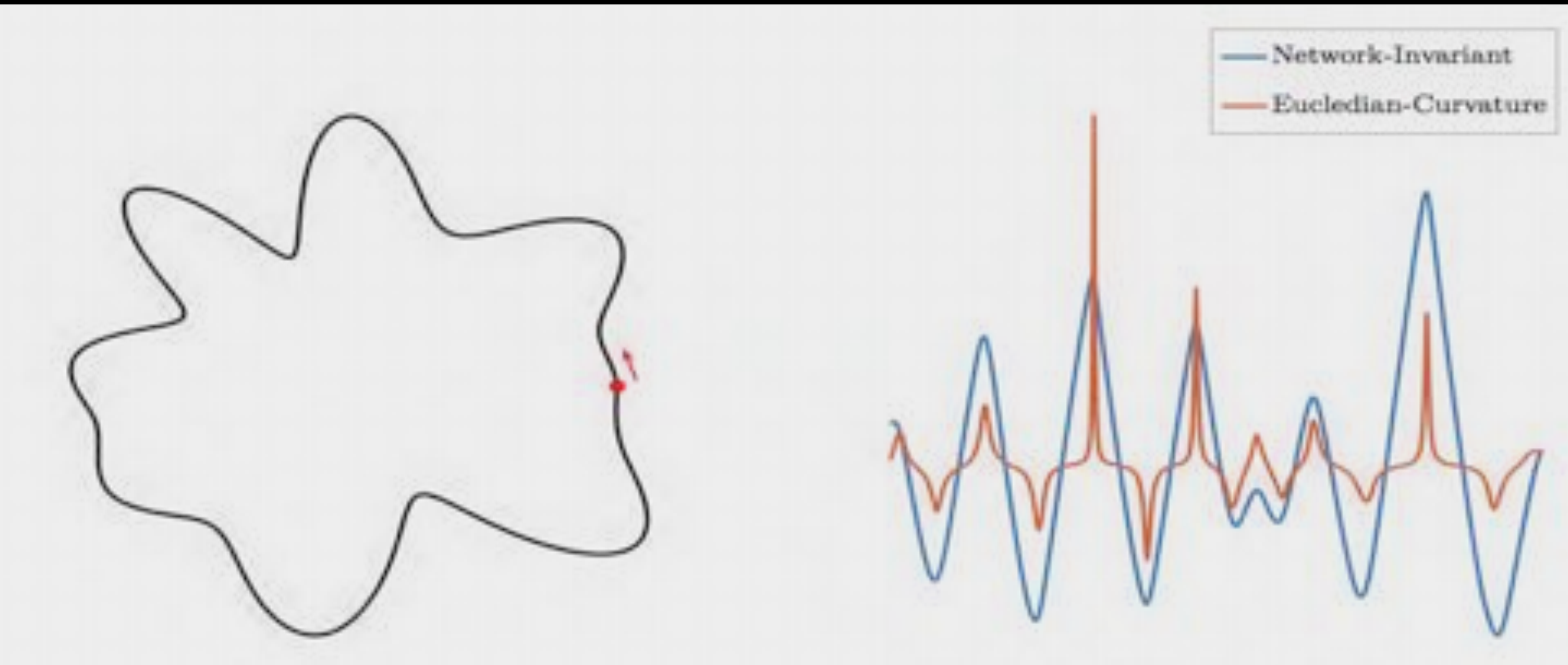
$\bar{g}_{ij} = (S_u, S_v, S_{ij})$

Equi-affine $g_{ij}^{EA} = \bar{g}_{ij} / \bar{g}^{1/4}$

Scale $\tilde{g}_{ij} = K g_{ij}$

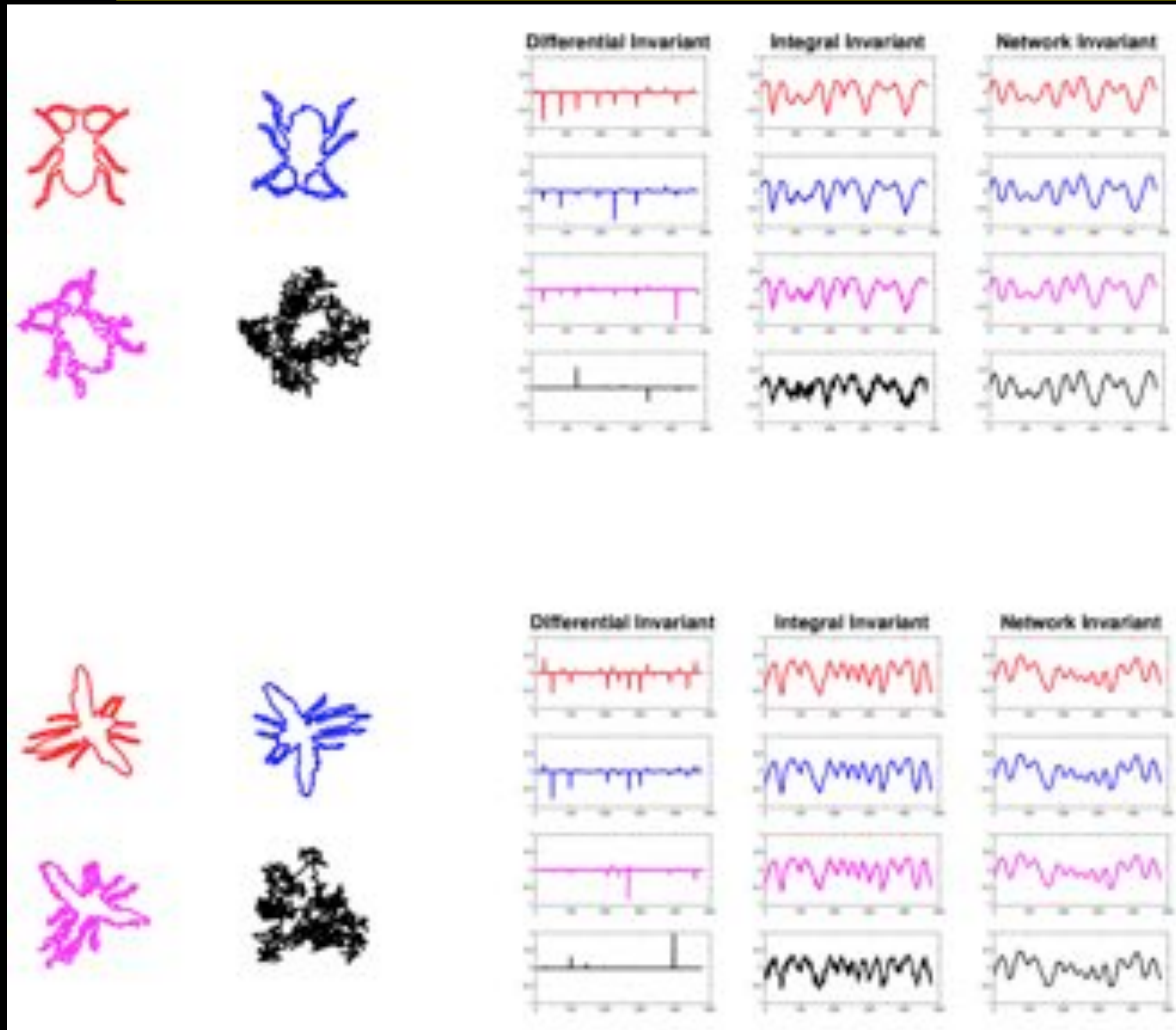
Affine $g_{ij}^{A} = K^{EA} g_{ij}^{EA}$

Learning invariants



Pai, Wetzler, Kimmel 2017

Robustness to noise



Pai, Wetzler, Kimmel 2017

Learning using Axiomatic Knowledge

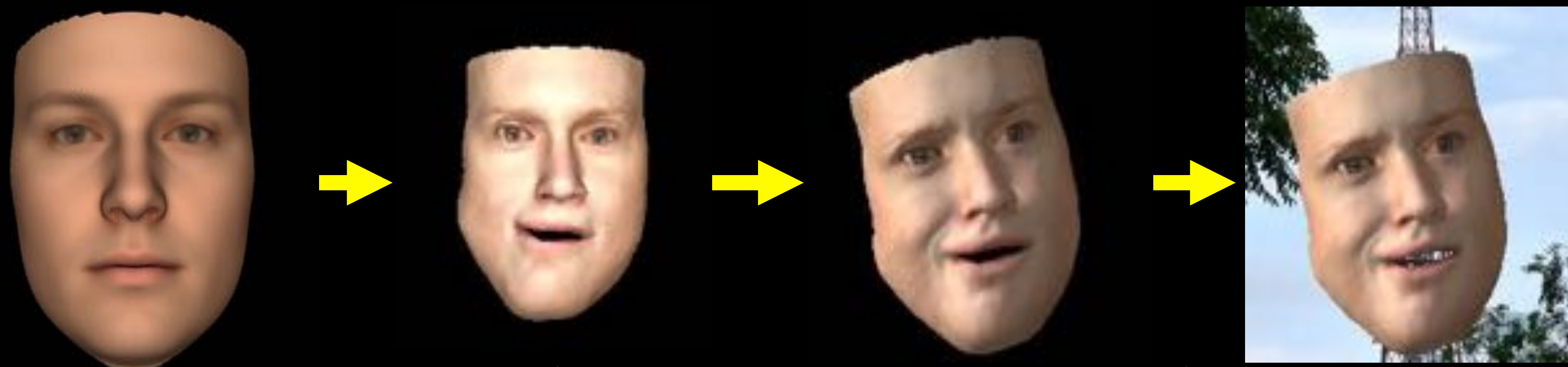
Richardson, Sela, Or-El, K. 2017

Learning using Axiomatic Knowledge

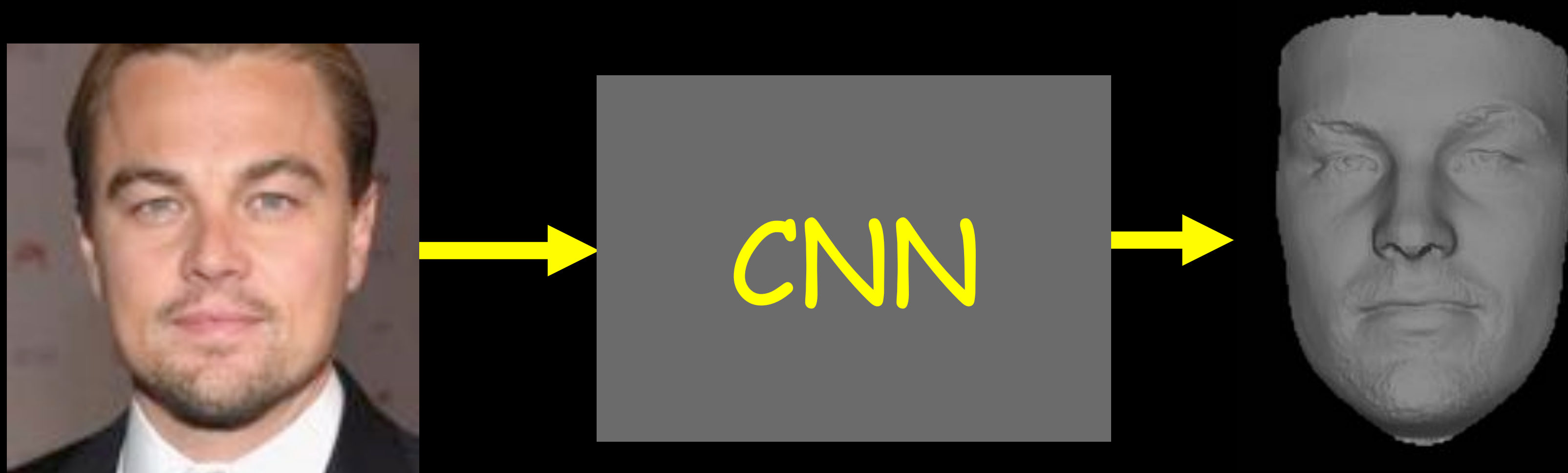
Richardson, Sela, Or-El, K. 2017

Learning using Axiomatic Knowledge

We know how to model faces



Can we use that to learn the inverse problem?



Richardson, Sela, Or-El, K. 2017


Thank you for
your attention
GIP