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Mathematical Perspectives on Large Eddy Simulation Models for Turbulent Flows

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Abstract. The main objective of this paper is to review and report on key mathematical issues related to the theory of Large Eddy Simulation of turbulent flows. We review several LES models for which we attempt to provide mathematical justifications. For instance, some filtering techniques and nonlinear viscosity models are found to be regularization techniques that transform the possibly ill-posed Navier–Stokes equation into a well-posed set of PDE's. Spectral eddy-viscosity methods are also considered. We show that these methods are not spectrally accurate, and, being quasi-linear, that they fail to be regularizations of the Navier–Stokes equations. We then propose a new spectral hyper-viscosity model that regularizes the Navier–Stokes equations while being spectrally accurate. We finally review scale-similarity models and two-scale subgrid viscosity models. A new energetically coherent scale-similarity model is proposed for which the filter does not require any commutation property nor solenoidality of the advection field. We also show that two-scale methods are mathematically justified in the sense that, when applied to linear non-coercive PDE's, they actually yield convergence in the graph norm.

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1. Introduction

1.1. Introductory comments

A common experience in everyday life is to observe the swirling motion of fluids flowing past objects or through conduits as so-called eddies are created and spun off to produce complicated flow patterns. Such phenomena are recognized as examples of turbulent flow, and an understanding of turbulence, its quantification, prediction, simulation, and control have become one of the most elusive and important goals in science and engineering.

It is now generally accepted that the Navier–Stokes equations, modeling the behavior of incompressible viscous fluids, describe accurately what is observed as

turbulence. Hence, considering that enormous computing power is available, one may be tempted to think that simulating numerically the Navier–Stokes equations should resolve the turbulence question. Unfortunately, despite steady advances in computing power, attempts at the Direct Numerical Simulation (DNS) of the Navier–Stokes equations have been limited to rather low values of the Reynolds number, R_e. At the present time, simulating time-dependent flows at Reynolds numbers greater than a few thousands is a daunting task. The reason for this very limited success of DNS is rooted in the heuristic Kolmogorov estimate $\mathcal{O}(R_e^{9/4})$ for the total number of degrees of freedom required to simulate flows at a given value of R_e. Considering the current pace of progress in computing power, this estimate undercuts the prospect of DNS of large-Reynolds number flows to some date possibly far in the future. Since the times of Reynolds and Boussinesq and others, approximations of the Navier–Stokes equations through the use of so-called turbulence models based on time-averaged or space-averaged quantities (Reynolds Averaged Navier–Stokes models, k-ϵ models, *etc.*) have been used in engineering applications as a means of overcoming, though often crudely, the formidable and, to date, virtually impossible task of DNS. The situation is further complicated by the absence of a complete mathematical theory of turbulence as described by either the Navier–Stokes equations directly or any of the various turbulence models, with the result that contemporary methods for studying such phenomena are often based on heuristics, empiricism, and mathematically unjustifiable assumptions.

In mathematical terms, the turbulence question is an elusive one. Since the bold definition of turbulence by Leray in the 1930's [47], calling turbulent solution any weak solution of the Navier–Stokes equations, progress has been frustratingly slow. The major obstacle in analyzing the Navier–Stokes equations has to do with the question of uniqueness of solutions, a question not yet solved owing to the possibility that the occurrence of so-called vorticity bursts reaching scales smaller than the Kolmogorov scale cannot be excluded.

In recent years, significant progress toward the development of useful turbulent models has occurred based on the observation that the whole range of flow scales may not be important in many significant engineering applications. In such applications, global information on "large-scale" features of the flow is sought for such physical quantities as momentum or internal energy. Hence, the notion that global behavior could be well approximated by a turbulence model without having to approximate accurately fine scales is viewed by many as a possible breakthrough in turbulence modeling. This has led modelers to devise artifacts for representing the interaction between the unreachable small scales and the large ones. These models are commonly known as Large Eddy Simulation (LES) models. Many LES models have been proposed, but no satisfactory mathematical theory for LES has yet been found.

1.2. Objectives of the paper

Our goal in the present paper is to report important mathematical results on LES modeling and to identify and elucidate several mathematical issues that stand in the way of developing a complete theory.

We review several classes of LES techniques for which we attempt to provide mathematical justifications. In particular, we show that some filtering techniques (*i.e* Leray mollifiers, NS-α model) and some nonlinear viscosity models (*i.e.* Smagorinsky) are indeed regularization techniques that transform the possibly ill-posed Navier–Stokes equations into a well-posed set of PDE's, thus giving strong mathematical support to these LES techniques.

Another significant aspect of LES theory, and one we shall address in this exposition, is the close relationship between mathematical properties of LES models and the numerical methods used to implement them in specific applications. This relationship was pointed by Ferziger [14]: "In general, there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated by many authors." In response to Ferziger's observation, we study the implementation of the Smagorinsky model; we conclude that, for the model to be mathematically coherent, the so-called "Smagorinsky constant" should not be constant, but should increase as the mesh is refined (or should take on values at least two to four times larger than what is usually recommended in the literature).

We also consider spectral eddy-viscosity methods. We show that this class of methods is closely related to the spectral viscosity methods introduced to solve nonlinear scalar conservation laws. Contrary to spectral viscosity methods, we show that eddy-viscosity methods cannot be spectrally accurate. Moreover, these techniques being quasi-linear, they are not strong enough to regularize the Navier–Stokes equation. We then propose a new class of spectral hyper-viscosity methods that regularize the Navier–Stokes equation while being spectrally accurate.

We finally review scale-similarity methods and two-scale subgrid viscosity techniques. We show that scale-similarity models are not energetically coherent and cannot be numerically implemented if the filter involved does not commute with differential operators and if the discrete vector fields are not exactly solenoidal. We propose a new scale-similarity model that is energetically coherent and whose implementation in weak form does not require the filter to satisfy the commutation property nor the advection fields to be solenoidal. The best mathematical justification we suggest for two-scale subgrid viscosity methods is that these techniques are very similar in their numerical implementation to another class of stabilization techniques that have been introduced to solve non-coercive PDE's and which are based on subgrid viscosity.

We finally conclude this paper by proposing mathematical criteria for selecting LES models.

1.3. Organization of the paper

The paper is organized as follows. In Section 2, we recall the energy cascade phenomenon and Kolmogorov scaling theory. We also cite some of the latest noteworthy mathematical results that rigorously corroborate this phenomenon. We then review in Section 3 the filtering method which stands out as the paradigm of LES for many authors. We point out that filtering may yield a paradox which, to our best knowledge, has not been recognized in the literature. We demonstrate also that filtering, if done correctly, falls into the class of regularization techniques that solve the uniqueness question. In Section 4, we study the Smagorinsky model. We show that this model belongs to the class of p-Laplacian regularizations and solves the uniqueness question as well. In Section 5, we analyze models based on spectral eddy-viscosity. We underline an analogy between these methods and the spectral viscosity methods which have been designed to be spectrally accurate and to guarantee convergence to the entropy solution when applied to scalar nonlinear conservation laws. We also propose a new spectral hyper-viscosity method which is spectrally accurate and regularizes the Navier–Stokes equations. In Section 6, we review models based on scale-similarity and two-level subgrid viscosity. We reformulate scale-similarity models in the framework of two-level approximation techniques and propose a new scale-similarity formulation preserving the energy pointwise. We finally show that two-level subgrid viscosity models are numerical methods that have stabilizing properties guaranteeing optimal convergence in some relevant graph norm when approximating non-coercive equations.

1.4. Notations and preliminaries

In this section we introduce notations that will be used throughout the paper. We also recall the definitions of standard functional spaces for the reader's convenience.

Unless explicitly stated, the fluid domain Ω is assumed to be an open bounded subset in $\mathbb{R}^3$ with a Lipschitz regular boundary Γ. When $\Omega = (0, 2\pi)^3$ and periodic boundary conditions are enforced in the three space directions, the domain Ω is referred to as the 3D-torus.

Real and complex-valued vectors/tensors are denoted in bold face. For any real and complex-valued vectors/tensors we denote by $|\cdot|$ the Hermitian norm, *i.e.* the ℓ^2-norm, and for any multi-index $\mathbf{k} \in \mathbb{Z}^d$ we set $|\mathbf{k}|_\infty = \max_{1\leq i\leq d} |k_i|$, *i.e.* the ℓ^∞-norm.

For $1 \leq p \leq +\infty$, we denote by $L^p(\Omega)$ the complex vector space of Lebesgue measurable functions such that

1) $\int_\Omega |f(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < +\infty$, if $1 \leq p < \infty$

2) $\inf\{M,\ |f(\mathbf{x})| < M \text{ for a.e. } \mathbf{x} \in \Omega\} < +\infty$, if $p = \infty$.

Partial derivatives of a function v with respect to variable ξ are denoted by $\partial_\xi v$; in the case in which v depends only on ξ, we will write $d_\xi v$. As usual, $W^{m,p}(\Omega)$

is the Sobolev space composed of functions that are in $L^p(\Omega)$ and whose partial derivatives up to order m are in $L^p(\Omega)$. We will use $H^m(\Omega)$ to refer to the space $W^{m,2}(\Omega)$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ composed of functions which vanish at the boundary. We shall denote by $\|\cdot\|_{m,p}$ the norm of $W^{m,p}(\Omega)$, making no distinction between the norms of scalar-valued and vector-valued function. When no confusion is possible we also denote by $\|\cdot\|_0$ the norm of $L^2(\Omega)$.

1.5. Navier–Stokes equations

Throughout this paper, we consider the Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } \Omega\times(0,T) \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \\ \mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{1.1}$$

where $\mathbf{u}_0$ is the initial data, $\mathbf{f}$ is a source term, and the density ρ is chosen equal to unity. The choice of no-slip or periodic boundary conditions simplifies somewhat the mathematical analysis without affecting the important features inherent to the Navier–Stokes equations. We also introduce two spaces of solenoidal vector fields:

$$\mathbf{V} = \{\mathbf{v}\in\mathbf{H}^1(\Omega),\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}_{|\Gamma}=0,\ \text{or } \mathbf{v} \text{ is periodic}\},$$
$$\mathbf{H} = \{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}\cdot\mathbf{n}_{|\Gamma}=0,\ \text{or } \mathbf{v} \text{ is periodic}\}.$$

The initial velocity $\mathbf{u}_0$ is assumed to be in $\mathbf{H}$. We denote by $\mathbf{P_H}$ the $\mathbf{L}^2$-orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$.

2. Energy cascade and Kolmogorov's scaling theory

In this section, we review Kolmogorov's scaling theory since it is very often referred to in LES, and, in order to provide a coherent introduction to this theory, we also recall the vortex stretching mechanism. In addition, we quote recently proven mathematical results which corroborate the energy cascade phenomenon in turbulent flows and discuss issues related to the existence of attractors for the Navier–Stokes dynamical system. For additional details on the material presented here we refer to Doering and Gibbon [12] and Foias *et al.* [19].

This section is meant to convince non-mathematician readers that, although the heuristic theory may give the feeling that everything is well understood, the mathematical theory is incomplete due to the uniqueness question, which is still open. Although this question is very often overlooked in practice, we show that it has important practical consequences.

Even though most of the physical arguments presented herein are standard to turbulence specialists, we feel it is important to bring them to the attention of

those more mathematically-minded readers who may not be acquainted with these notions.

2.1. The vortex stretching mechanism

We first rewrite the momentum equation in terms of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ by taking the curl of the momentum equation

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} + \mathbf{D} \cdot \boldsymbol{\omega} + \nabla \times \mathbf{f}, \tag{2.1}$$

where $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the deformation tensor (also called strain rate tensor). The evolution equation for $\boldsymbol{\omega}$ resembles in many aspects the momentum equation governing the velocity field $\mathbf{u}$, except for one term, the so-called *vortex stretching term* $\mathbf{D} \cdot \boldsymbol{\omega}$. This term explains some fundamental differences between flows in two or three dimensions.

In two space dimensions, for instance in the (x, y) plane, the velocity $\mathbf{u}$ has components $(u, v, 0)$ which implies that the vorticity has a nonzero component only in the z-direction, *i.e.*, $\boldsymbol{\omega} = (0, 0, \omega)$. It immediately follows that the vortex stretching term $\mathbf{D} \cdot \boldsymbol{\omega}$ vanishes identically and does not contribute to the evolution of the vorticity field.

In three space dimensions, the term $\mathbf{D} \cdot \boldsymbol{\omega}$ does not necessarily vanish and may give rise to a potentially strong local phenomenon referred to as the *vortex stretching mechanism* (a mechanism held responsible for intensifying the circulation in tornadoes). Recall that $\mathbf{D}$ is symmetric, hence it is diagonalizable; furthermore, the trace of $\mathbf{D}$ being zero ($\mathrm{tr}(\mathbf{D}) = \nabla \cdot \mathbf{u} = 0$), $\mathbf{D}$ has at least one non-negative eigenvalue. If it happens that $\boldsymbol{\omega}$ is aligned with one eigenvector associated with a positive eigenvalue of $\mathbf{D}$, then the vorticity magnitude and the angular velocity increase as long as the diffusive term $\nu \nabla^2 \boldsymbol{\omega}$ and the source term $\nabla \times \mathbf{f}$ are not strong enough to counterbalance this mechanism. From a physical point of view, this implies that a fluid element would first contract in the direction perpendicular to the vorticity vector, and then stretch along this direction in order for the angular momentum to be conserved (if dissipation is neglected). The vortex stretching mechanism is held responsible for local amplification of the vorticity magnitude and thus for the production of smaller and smaller scale structures in the flow field. This phenomenon thus implies a transfer of energy from large length scales to smaller ones, usually known as the energy cascade.

2.2. Fourier analysis and energy cascade

The study of homogeneous isotropic turbulence is usually performed on the periodic three-dimensional domain $\Omega = (0, L)^3$ using spectral analysis. Instead of considering the 3D torus $(0, 2\pi)^3$, we prefer to keep the dimension L as we will subsequently use dimensional analysis. Since $\mathbf{u}(\mathbf{x}, t)$ is square integrable in space,

the velocity field can be expanded in terms of the Fourier series

$$\mathbf{u}(\mathbf{x},t)=\sum_{\boldsymbol{\ell}}\hat{\mathbf{u}}(\boldsymbol{\ell},t)\,e^{i\boldsymbol{\ell}\cdot\mathbf{x}},$$

where the wavenumbers $\boldsymbol{\ell}$ are given by $\boldsymbol{\ell}=2\pi\boldsymbol{n}/L$, $\boldsymbol{n}\in\mathbb{Z}^3$, and where the Fourier coefficients, or modes, satisfy:

$$\hat{\mathbf{u}}(\boldsymbol{\ell},t)=\frac{1}{L^3}\int_\Omega\mathbf{u}(\mathbf{x},t)\,e^{-i\boldsymbol{\ell}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}.$$

The main advantage of such a decomposition is that it permits to distinguish the various length scales in the velocity field, the length scale associated with the wavenumber $\boldsymbol{\ell}$ being defined, for example, as $2\pi/|\boldsymbol{\ell}|_\infty$.

Taking the Fourier Transform of the Navier–Stokes equations, we can derive the time evolution of each mode $\hat{\mathbf{u}}(\boldsymbol{\ell},t)$:

$$d_t\hat{\mathbf{u}}(\boldsymbol{\ell},t)+\nu|\boldsymbol{\ell}|^2\hat{\mathbf{u}}(\boldsymbol{\ell},t)=\left(\mathbf{I}-\frac{\boldsymbol{\ell}\boldsymbol{\ell}^T}{|\boldsymbol{\ell}|^2}\right)\cdot\Big[\hat{\mathbf{f}}(\boldsymbol{\ell})-i\sum_{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2=\boldsymbol{\ell}}[(\hat{\mathbf{u}}(\boldsymbol{\ell}_1,t)\cdot\boldsymbol{\ell}_2)\hat{\mathbf{u}}(\boldsymbol{\ell}_2,t)]\Big]\quad(2.2)$$

where $\mathbf{I}$ is the unit tensor, $\mathbf{I}-\boldsymbol{\ell}\boldsymbol{\ell}^T/|\boldsymbol{\ell}|^2$ the projector onto divergence free vector fields in wavenumber space, and $\hat{\mathbf{f}}(\boldsymbol{\ell})$ the Fourier modes of the body force, assuming $\mathbf{f}$ to be independent of time. The body force term $\hat{\mathbf{f}}(\boldsymbol{\ell})$ supplies the energy to the system at wavenumber $\boldsymbol{\ell}$ in the support of $\hat{\mathbf{f}}$. We suppose here that it is only significant for long wavelengths and provides for sufficient power to maintain the system within a permanent turbulent regime. This term does not directly excite the short scales present in the flow. The second term in (2.2) is the so-called *viscous dissipation term* $\nu|\boldsymbol{\ell}|^2\hat{\mathbf{u}}(\boldsymbol{\ell},t)$. Because of the factor $|\boldsymbol{\ell}|^2$, viscous dissipation is more effective at short rather than at large length scales. The last term is the result of the Fourier transform of the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$. This term actually allows for mode coupling in the wavenumber space, a mechanism which provides for the activation of shorter and shorter length scales in the flow. Equation (2.2) clearly shows that any triad $(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)$ are coupled if and only if one of these wavenumbers is the sum of the other two. In other words, the nonlinear term allows for the transfer of energy from the large scales, excited by the body force, to the smallest scales, for which viscous dissipation becomes predominant. This mechanism is usually referred to as the energy cascade.

The energy cascade was perhaps intuitively imagined by Leonardo da Vinci as early as the sixteen century when he wrote: "...the small eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by both small eddies and large." An alternative description of the energy cascade is given by Lesieur [48] in the following terms: the flow reaches an equilibrium state where the vortex stretching mechanism produces an "infinite hierarchy of eddies; each of them sucking the energy of the bigger ones on which they ride, while they are being sucked by the smaller eddies riding on them" the feast stopping at the viscous dissipation scale.

Note that the nonlinear term does not participate to the global kinetic energy balance since

$$\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}^2)\,\mathrm{d}\mathbf{x} = \int_\Omega \nabla\cdot(\tfrac{1}{2}\mathbf{u}^2\mathbf{u})\,\mathrm{d}\mathbf{x} = 0. \tag{2.3}$$

Once again, it is clear that the role of the nonlinear term is to redistribute the energy from the large scales to the small ones.

We will refer to this energy cascade mechanism again in the next section to present Kolmogorov's scaling theory. Meanwhile, we demonstrate how to decompose the kinetic energy of the flow field into contributions from each wavenumber or length scale. Indeed, after making use of Parseval's equality, the instantaneous mean value of the kinetic energy is given by

$$\frac{1}{L^3}\int_\Omega \frac{1}{2}\mathbf{u}^2(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \sum_{\boldsymbol{\ell}} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 = \sum_k \Big\{ \sum_{|\boldsymbol{\ell}|_\infty = k} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 \Big\}$$

where $k = 2\pi n/L$, $n \in \mathbb{N}$. A possible decomposition of the kinetic energy is therefore suggested as

$$\frac{1}{2L^3}\|\mathbf{u}(\cdot,t)\|_0^2 = \frac{2\pi}{L}\sum_k E(k,t)$$

where the quantity

$$E(k,t) := \frac{L}{2\pi}\sum_{|\boldsymbol{\ell}|_\infty = k} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 \tag{2.4}$$

defines the kinetic energy associated with wavenumbers $\boldsymbol{\ell}$ such that $|\boldsymbol{\ell}|_\infty = k$, or equivalently, associated with the length scale $2\pi/k$. Note that we have used the ℓ^∞ norm to define scalar wavenumbers k. We could also have used the ℓ^2 norm as it is commonly done in the literature, but this requires to define spherical shells in the spectral space and complicates unnecessarily the presentation without bringing forth any new fundamental idea. We also refer to §2.4. where an alternative definition of wavenumbers is used.

2.3. Kolmogorov's scaling theory

When a homogeneous isotropic flow reaches a permanent steady-state regime, the energy cascade mechanism suggests the following hypothesis:

> *The rate of energy cascading from the largest scales to the finest ones is assumed to be constant and independent of time.* (HK)

This hypothesis seems reasonable because the energy is supplied to the system by the body force at large length scales, and is consumed by viscous dissipation at the short scales. However, due to (2.3), we observe that the average energy input

rate is the same as the average energy dissipation rate, which we denote by ϵ:

$$\epsilon = \frac{\nu}{L^3}\langle \|\nabla \mathbf{u}\|_0^2 \rangle, \tag{2.5}$$

where $\langle\cdot\rangle$ denotes the time average, and where it is assumed that $\langle \|\nabla \mathbf{u}\|_0^2 \rangle$ exists. In the intermediate range, the energy is transferred by nonlinear effects. Supposing that viscous effects are negligible for these scales, it follows that the energy $E(k)$ should depend only of k and ϵ and not directly on the viscosity; supposing that it assumes the form

$$E(k) \sim \epsilon^a k^b$$

and invoking the dimensional relations (L for length and T for time)

$$[k] = [L]^{-1}, \quad [\epsilon] = [L]^2[T]^{-3}, \quad [E(k)] = [L]^3[T]^{-2},$$

we obtain $a = \frac{2}{3}$ and $b = -\frac{5}{3}$, giving the well-known formula:

$$E(k) = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{2.6}$$

where C_K is believed to be a "universal" dimensionless constant.

We now derive the Kolmogorov length scale λ_K, defined as the scale at which inertial effects actually balance viscous dissipation. The length scale λ_K can be viewed as the smallest active scale in the flow, or simply, as the smallest energetically relevant scale. Since the Reynolds number measures the energetic ratio between inertial terms and viscous terms, the Kolmogorov length scale can be determined by: $U_{k_K}\lambda_K/\nu \sim 1$, where U_{k_K} is the velocity scale associated with $k_K = 2\pi/\lambda_K$ and is obtained by using the scaling $U_{k_K} = (k_K E(k_K))^{1/2}$. Then, by using relation (2.6) we obtain

$$\lambda_K = c_K \nu^{3/4} \epsilon^{-1/4}, \tag{2.7}$$

where c_K is a dimensionless constant which can be related to C_K (see *e.g.* [12, p. 55]).

Finally, we relate the Kolmogorov length scale to a global Reynolds number. As for U_{k_K}, we first define a macroscopic velocity scale U as

$$\frac{1}{2}U^2 := \frac{2\pi}{L}\sum_{k=2\pi/L}^{2\pi/\lambda_K} E(k).$$

Using (2.6), we observe that this sum is approximately equal to $\epsilon^{2/3}L^{2/3}$, which yields:

$$\epsilon \sim L^{-1}U^3. \tag{2.8}$$

Now, from the definition of the Reynolds number $R_e = UL/\nu$ and by using (2.8) and (2.7) we infer:

$$\frac{\lambda_K}{L} \sim R_e^{-3/4}. \tag{2.9}$$

This relation is widely used in computational fluid mechanics to estimate the number of cells needed to fully resolve turbulent flows. For example, if one is

interested in simulating a turbulent flow at $R_e = 1000$ in a unit cube domain ($L = 1$) using finite difference or finite element methods, one would need to consider grid sizes of order $\lambda_K/L \sim R_e^{-3/4} \sim 5.6 \times 10^{-3}$ in each direction, that is, approximately $(L/\lambda_K)^3 \sim R_e^{9/4} \sim 5.6$ million cells in total.

Kolmogorov's scaling argument is both a simple and a surprising result. Astonishingly, the argument does not take into account the complex structure of the nonlinear term of the Navier–Stokes equations. Actually "Kolmogorov's similarity theory does not rely on the Navier–Stokes equations. Kolmogorov [40] did not even mention them." (see Muschinski [59, p. 240]). It is merely based on an intuitive interpretation of the vortex stretching and energy cascade mechanisms. However, it is noteworthy that the scaling law (2.6) has been actually observed in numerous laboratory experiments, especially in the small scale range (see *e.g.* Kolmogorov [39], Lesieur [48, p. 87], or the review of Sreenivasan [70]). It is even more remarkable that a mathematical result, proposed by Foias *et al.* [18] and presented below, corroborates the correctness of Kolmogorov's scaling theory.

2.4. Mathematical justification of Kolmogorov's scaling theory

A quite simple mathematical justification of the energy cascade mechanism and hence of the Kolmogorov's scaling theory has been recently proposed by Foias *et al.* in [18]. We summarize here the principal ideas.

The authors consider either periodic boundary conditions in the 3D-torus or no-slip boundary conditions, assuming in the second case that Γ is $\mathcal{C}^2$. They introduce the Stokes operator $A = -\mathbf{P_H}\nabla^2 : \mathbf{V} \cap \mathbf{H}^2(\Omega) \to \mathbf{H}$. Being positive and self-adjoint, and its inverse compact, the Stokes operator possesses a basis of eigenvectors $(\mathbf{w}_j)_{j\geq 0}$, which is orthonormal and complete in $\mathbf{H}$, and the corresponding eigenvalues are such that $0 < \lambda_0 \leq \lambda_1 \leq \ldots \lambda_j \to +\infty$, as $j \to \infty$. In this basis, any weak solution of the Navier–Stokes equations can be expanded as

$$\mathbf{u} = \sum_{j=0}^{+\infty} \hat{u}_j \mathbf{w}_j.$$

At this stage of the exposition, it is convenient to introduce the following notations:

$$k_j := \lambda_j^{1/2} \qquad \mathbf{u}_k := \sum_{\lambda_j = k^2} \hat{u}_j \mathbf{w}_j \qquad \mathbf{u}_{k',k''} := \sum_{k' \leq k < k''} \mathbf{u}_k.$$

In other words, k_j is the wavenumber associated with the eigenvalue λ_j; $\mathbf{u}_k$ is the sum of all components of $\mathbf{u}$ that are indexed by the same wavenumber k; $\mathbf{u}_{k',k''}$ is the partial expansion of $\mathbf{u}$ between wavenumbers k' and k''.

The Navier–Stokes equations (1.1) then reduce to

$$\begin{cases} \partial_t \mathbf{u}_k + \nu A \mathbf{u}_k + (\mathbf{P_H}(\mathbf{u} \cdot \nabla \mathbf{u}))_k = \mathbf{f}_k, \quad \forall\ k \geq k_0, \\ \mathbf{u}_{k|t=0} = \mathbf{u}_{0,k}, \end{cases} \tag{2.10}$$

where the source term $\mathbf{f}$ is assumed to be time independent, to belong to $\mathbf{H}$, and to be localized in the spectral space: *i.e.*, it is assumed that there exist $\underline{k}_f$ and $\overline{k}_f$ such that $0 < \underline{k}_f \leq \overline{k}_f < +\infty$ and $\mathbf{f} = \mathbf{f}_{\underline{k}_f, \overline{k}_f}$. We also introduce the medium wavenumber k_f as

$$k_f = \frac{\|A^{1/2}\mathbf{f}\|_0}{\|A^{-1/2}\mathbf{f}\|_0}$$

which naturally satisfies $\underline{k}_f \leq k_f \leq \overline{k}_f$.

One of the key ingredients now consists in defining the time average of quantities of interest (kinetic energy, enstrophy, *etc.*). Being given a function of time $g(t)$, our goal here is to give a sense to the limit:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\tau)\,\mathrm{d}\tau$$

for it may happen that this limit does not exist (think of $g(t) = \sin(\log(1+t)) + \cos(\log(1+t))$). Let G be the set of bounded real-valued functions on $[0, +\infty)$ and denote by G_0 the subset of G whose functions admit a limit at $+\infty$. Define $p : G \longrightarrow \mathbb{R}$ such that $p(g) = \limsup_{t\to+\infty} g(t)$ and $\Lambda_0 : G_0 \longrightarrow \mathbb{R}$ such that $\Lambda_0(g) = \lim_{t\to+\infty} g(t)$. It is clear that for all g and f in G, $p(g+f) \leq p(g) + p(f)$ and $p(\gamma g) = \gamma p(g)$ for all $\gamma > 0$. Furthermore, for all g in G we have $\Lambda_0(g) \leq p(g)$; as a result, owing to the Hahn–Banach theorem, there exists $\Lambda : G \longrightarrow \mathbb{R}$, such that $\Lambda(g) = \Lambda_0(g)$ for all g in G_0 and for all g in G $\Lambda(g) \leq p(g)$; *i.e.* Λ is an extension of Λ_0 (see [19, p. 225] and [17] for more details). The operator Λ is the generalized limit and we hereafter denote $\Lambda(g) = \mathrm{LIM}_{t\to+\infty}\, g(t)$. Equipped with this new tool, it is now possible to define time-averages for any function g in G by

$$\langle g \rangle = \operatorname*{LIM}_{t\to+\infty} \frac{1}{t}\int_0^t g(\tau)\,\mathrm{d}\tau. \tag{2.11}$$

Actually, this definition can be broadened even more as shown in [19, p. 191].

Now we can define the average kinetic energy $e = k_0^3\langle\|\mathbf{u}\|_0^2\rangle/2$ and the average dissipation rate per unit time and mass $\epsilon = \nu k_0^3\langle\|\nabla\times\mathbf{u}\|_0^2\rangle$. We introduce $k_K = (\epsilon/\nu^3)^{1/4}$ and $k_T = (\epsilon/2\nu e)^{1/2}$, the Kolmogorov and Taylor wavenumbers, respectively. Introducing the characteristic length $L_0 = k_0^{-1}$ and the characteristic velocity U such that $e = U^2/2$, we can define the Reynolds number $R_e = UL_0/\nu$. Using standard energy estimates, it is easy to derive the following bound:

$$R_e = \frac{k_0^{1/2}\langle\|\mathbf{u}\|_0^2\rangle^{1/2}}{\nu} \leq \frac{\langle\|A^{1/2}\mathbf{u}\|_0^2\rangle^{1/2}}{\nu k_0^{1/2}} \leq \frac{\|A^{-1/2}\mathbf{f}\|_0}{\nu^2 k_0^{1/2}}. \tag{2.12}$$

Then, the following theorem is proved in [18]:

Theorem 2.1. *Provided R_e is sufficiently large so that $R_e \geq (k_0/k_f)^{1/2}$, there exists a constant c such that*

$$\epsilon \leq ck_0U^3, \quad k_K \leq ck_0R_e^{3/4}, \quad k_T \leq ck_0R_e^{1/2}.$$

This theorem is a rigorous justification, though still incomplete, of the estimates on ϵ and k_K established in (2.8) and (2.9) and of another standard estimate on k_T that can be obtained by Kolmogorov's scaling theory.

The most striking result from [18] deals with energy fluxes. Taking the inner product of (2.10) with $\mathbf{u}_{k'k''}$ in $\mathbf{H}$, taking the sum on k from some k' to some $k'' > k'$, yields

$$\tfrac{1}{2} d_t \|\mathbf{u}_{k',k''}\|_0^2 + \nu \|\nabla \mathbf{u}_{k',k''}\|_0^2 = (\mathbf{f}_{k',k''}, \mathbf{u}_{k',k''}) + e_{k'}(\mathbf{u}) - e_{k''}(\mathbf{u}), \tag{2.13}$$

where we have set

$$e_k(\mathbf{u}) = (\mathbf{u}_{k,\infty} \cdot \nabla \mathbf{u}_{k,\infty}, \mathbf{u}_{k_0,k}) - (\mathbf{u}_{k_0,k} \cdot \nabla \mathbf{u}_{k_0,k}, \mathbf{u}_{k,\infty}).$$

The quantity $e_k(\mathbf{u})$ can be interpreted physically as being the flux per unit time of kinetic energy into the higher modes $\mathbf{u}_{k,\infty}$ induced by vortex stretching. Taking the time average of (2.13) and assuming $k'' > k' > \overline{k}_f$, we deduce

$$\langle e_{k'}(\mathbf{u}) \rangle - \langle e_{k''}(\mathbf{u}) \rangle = \nu \|\nabla \mathbf{u}_{k',k''}\|_0^2. \tag{2.14}$$

Modulo some technical details pertaining to the possible loss of regularity as $k'' \to +\infty$ (the technical issue that is responsible for the uniqueness question), it is possible to pass to the limit $k'' \to +\infty$, so that equality (2.14) is replaced by the inequality

$$\langle e_{k'}(\mathbf{u}) \rangle \geq \nu \|\nabla \mathbf{u}_{k',\infty}\|_0^2. \tag{2.15}$$

As a result, we conclude from (2.14)–(2.15) that for $k > \overline{k}_f$, the energy flux $\langle e_k(\mathbf{u}) \rangle$ is nonnegative and monotone decreasing. Hence, energy flows down from the scales enforced by the forcing term to the small scales, as suggested by the arguments on vortex stretching. From (2.14) it is now easy to derive the following bounds:

Theorem 2.2 (Foias *et al.* [18]). *Provided $k'' > k' > \overline{k}_f$, we have*

$$0 \leq 1 - \frac{\langle e_{k''}(\mathbf{u}) \rangle}{\langle e_{k'}(\mathbf{u}) \rangle} \leq \left(\frac{k''}{k_T} \right)^2 \left(1 - \left(\frac{k'}{k_T} \right)^2 \right)^{-1}$$

and, provided $k > \overline{k}_f$, then

$$1 - \left(\frac{k}{k_T} \right)^2 \leq \frac{k_0^3 \nu \langle \|\nabla \mathbf{u}_{k,\infty}\|_0^2 \rangle}{\epsilon} \leq 1.$$

The two bounds established in this theorem have the following physical interpretation. The first one implies that

$$\text{if } k_T \gg k'' > k' > \overline{k}_f, \text{ then } \langle e_{k''}(\mathbf{u}) \rangle \approx \langle e_{k'}(\mathbf{u}) \rangle;$$

that is, the energy flux through wavenumbers k within the range $k_T \gg k > \overline{k}_f$ is nearly constant, confirming the energy cascade scenario as imagined by Kolmogorov. The second bound yields

$$\text{if } k_T \gg k > \overline{k}_f, \text{ then } k_0^3 \nu \langle \|\nabla \mathbf{u}_{k,\infty}\|_0^2 \rangle \approx \epsilon;$$

that is, for scales such that the corresponding wavenumbers k are within the range $k_T \gg k > \overline{k}_f$, the energy lost by viscous dissipation below these scales is nearly independent of k and approximately equal to the energy dissipation rate ϵ, confirming the well-known hypothesis by Kolmogorov stated in (HK).

2.5. Attractors of the Navier–Stokes equations

The attractors of autonomous dissipative dynamical systems characterize the long time behavior of flows and are closely related to their stability properties. The attractor is a compact subset of the phase space (the normed space $\boldsymbol{L}^2(\Omega)$ here) toward which flow solutions converge after long time periods. Although there exist various definitions of the dimension of a set, an upper bound on the dimension of the attractor is obtained by taking the smallest N for which all N-dimensional subsets of $\boldsymbol{L}^2(\Omega)$ (sets of initial states at $t = 0$ for example) contract to zero volume as $t \to \infty$ (see [12, p. 170–171] or [19, p. 117–118] for more details). For example, the dimension of a steady-state flow (fixed point) is zero while the dimension of a periodic flow (periodic orbit) is one. For more complex flows, the attractor dimension is not necessarily an integer and the estimate (2.9) from Kolmogorov's scaling theory yields the widely accepted conjecture that the Navier–Stokes equations should have a global attractor of dimension $d \leq \mathcal{O}(R_e^{9/4})$.

However, the question of whether a global attractor of the Navier–Stokes equations in 3D exists is an issue that is not yet settled. This question has important practical implications, for knowing that an attractor exists and that this attractor is finite dimensional would guarantee that long-time behavior of Navier–Stokes solutions can reasonably be numerically approximated (*i.e.* using a finite number of degrees of freedom).

The question of the existence of an attractor is intimately linked with the problem of knowing whether time-dependent Navier–Stokes solutions are unique, or, equivalently, of ascertaining that the time evolution governing the solutions is deterministic. As surprising as it may be, this simple question is still open. The main obstacle in the way is that it cannot be yet proven that the solutions to the 3D Navier–Stokes equations are smooth for arbitrarily long times. Up to now, no *a priori* estimate has been found that guarantees that the enstrophy, usually defined as $\|\nabla \times \mathbf{u}\|_0^2$, remains finite for all times; that is, no *a priori* estimate guarantees that the vorticity does not blow up somewhere in the domain in finite time. Note, however, that for a given forcing term $\mathbf{f}$ and a given initial data $\mathbf{u}_0$, the quantity $\frac{1}{t}\int_0^t \|\nabla \times \mathbf{u}\|_0^2 \, \mathrm{d}\tau$ is bounded; hence, the enstrophy is bounded in the mean. Nevertheless, this bound does not preclude the enstrophy to blow-up intermittently like $(t - t_0)^\alpha$ with $-1 < \alpha < 0$. Hence, we cannot exclude *a priori* the possible occurrence of rare (intermittent) vorticity bursts driving the energy deep down to scales much shorter than the standard Kolmogorov scale. At the present state of the art, one cannot prove or disprove mathematically that

the energy cascade scenario does indeed stop at the Kolmogorov scale everywhere in the domain. In other words, in the present level of understanding, we cannot disprove the possibility that the nonlinear term (the vortex stretching mechanism) may be so strong that one is not guaranteed that the linear viscosity is strong enough to stop the cascade everywhere in the fluid domain.

If such a blow-up were to occur, the time-evolution of the solution would not be unique; this would be unacceptable as deterministic Newtonian time evolution would be lost. Moreover, finite-time singularities would mean that the flow would develop arbitrarily small-scale structures, violating the axiom of continuum mechanics assuming a scale separation between individual atomic evolution and collective hydrodynamics motions. In conclusion, though the uniqueness question may seem to be an irrelevant issue, it is "actually intimately tied up with the efficiency of the Navier–Stokes equations as a model for fluid turbulence" [12, p. xii].

As a result, information on the attractor, or equivalently on the smallest active scale of the flow, can be obtained only by assuming *a priori* some regularity on the enstrophy. For instance, in the 3D-torus we have the following result proved in Constantin *et al.* [11]:

Theorem 2.3. *Assuming that the quantity*

$$\epsilon_1 = \limsup_{t\to+\infty} \sup_{\mathbf{u}_0\in\mathbf{H}} \frac{1}{t}\int_0^t \nu\|\nabla\times\mathbf{u}(\tau,\cdot)\|_{0,\infty}^2\,\mathrm{d}\tau,$$

is finite and defining the corresponding Kolmogorov scale $\lambda_1 = \nu^{3/4}\epsilon_1^{-1/4}$*, then the Lyapunov dimension,* d_L*, of the global attractor of the 3D Navier–Stokes equations is bounded from above as follows:*

$$d_L \le c\left(\frac{L}{\lambda_1}\right)^3. \tag{2.16}$$

Note that the fractal dimension d_F (box counting) and the Hausdorff dimension d_H are such that $d_H \le d_F \le d_L$, so the estimate (2.16) is actually a bound on these three dimensions. This bound shows that the Kolmogorov scale, λ_1, is the smallest active scale of the flow if ϵ_1 is finite.

Any attempt trying to weaken the smoothness hypotheses on the vorticity has so far yielded suboptimal results. For instance, we have the following result proved by Gibbon and Titi [24].

Theorem 2.4. *Assuming that* $\epsilon_2 = \nu L^{-3}\sup_t\|\nabla\times\mathbf{u}\|_{0,2}^2$ *is finite and defining the corresponding Kolmogorov scale* $\lambda_2 = \nu^{3/4}\epsilon_2^{-1/4}$*, we have the following bound on* d_L*:*

$$d_L \le c\left(\frac{L}{\lambda_2}\right)^{4.8}. \tag{2.17}$$

Hence, assuming that the enstrophy is finite for all times, the above estimate shows that the smallest length scale ℓ of the flow is

$$\frac{\ell}{L} \geq c \left(\frac{\lambda_2}{L} \right)^{1.6}.$$

In other words, weakening the assumption on the regularity of the vorticity translates immediately into the possible existence of active scales, ℓ, smaller than the Kolmogorov scale λ_2.

We conclude this section by recalling a consequence of the ladder theorem of Doering *et al.* (see [3] and [12, p. 143]): if for some $\delta > 0$, $\sup_t \|u\|_{0,3+\delta}$ is finite then no singularity can occur. The reader will note that the gap between what is known to be bounded, *i.e.* $\sup_t \|u\|_{0,2}$, and what should be bounded for the whole Navier–Stokes analysis to fall neatly in place, $\sup_t \|u\|_{0,3+\delta}$, is frustratingly small.

3. The filtering concept in LES

Large Eddy Simulation, or simply LES, an acronym coined in the ground breaking paper of Leonard [46], has as its primary goal to modify the Navier–Stokes equations in order to obtain a new system of equations which is more amenable to approximate while retaining all the most energetic features of the unperturbed problem. The classical idea is to use a filter which allows for the separation of large and small length scales in the flow-field. Applying the filtering operator to the Navier–Stokes equations provides a new equation governing the large scales, except for one term involving the small velocity scales. Modeling this term in an appropriate manner, a procedure commonly referred to as the *closure problem*, one can arrive at a set of equations with only the large velocity (and pressure) scales as the unknown. In this section, we first show that the closure problem associated with the filtering procedure actually gives way to a paradox. In addition, we show another interesting result establishing that adequate filtering regularizes the Navier–Stokes equations.

This section is organized as follows. First we introduce the filtering concept and the closure problem; then we show that exact LES is possible but paradoxical. In a third section, we show that the filtering technique introduced by Leray [47] in the 1930's is a good mathematical justification for LES in the sense that it yields a well-posed problem. In a fourth section, we show that the Leray regularization can be modified to be frame indifferent and we discover a link with the so-called Navier–Stokes$-\alpha$ model [9]. In the last section, using a result by Duchon and Robert [13, 63], we show that LES solutions computed by means of the Leray regularization satisfy an energy inequality pointwise that Navier–Stokes solutions computed without LES model might violate, thus giving another justification for filter-based LES.

The main conclusions of this investigation can be summarized as follows:

(i) Exact LES is possible but paradoxical.

(ii) Filtering *à la* Leray solves the uniqueness question, thus justifying filtering as a mean to do LES.
(iii) Filtering *à la* Leray selects "dissipative" Navier–Stokes solutions.

3.1. Filtering operator and closure problem

Let us assume that we have at hand an operator $\overline{(\cdot)} : w \longmapsto \overline{w}$ with filtering properties either in space or in time, or both in space and time. For the time being, avoiding to be too specific about the nature of this filter, we simply assume that the operator is linear and commutes with differential operators. For the purpose of the present discussion, these properties are shared by most filters of interest when Ω is the 3D torus. For examples of filters we refer the reader to [65].

Applying the filtering operator to the Navier–Stokes equations yields the new system of equations:

$$\begin{cases} \partial_t \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + \nabla \overline{p} - \nu \nabla^2 \overline{\mathbf{u}} = \overline{\mathbf{f}} - \nabla \cdot \mathbb{T}, \\ \nabla \cdot \overline{\mathbf{u}} = 0 \\ \overline{\mathbf{u}}|_\Gamma = 0, \quad \text{or } \overline{\mathbf{u}} \text{ is periodic}, \\ \overline{\mathbf{u}}|_{t=0} = \overline{\mathbf{u}}_0, \end{cases} \tag{3.1}$$

where

$$\mathbb{T} = \overline{\mathbf{u} \otimes \mathbf{u}} - \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} \tag{3.2}$$

is usually referred to as the subgrid-scale tensor. In order to be able to solve (3.1) for $\overline{\mathbf{u}}$ without having to resort to $\mathbf{u}$, *i.e.* to close the problem, the tensor $\mathbb{T}$ needs to be expressed in term of $\overline{\mathbf{u}}$ only. The closure problem which consists of finding an accurate model $\mathbb{T}(\overline{\mathbf{u}})$ for the subgrid-scale tensor $\mathbb{T}$ certainly represents the main difficulty of LES.

3.2. The closure paradox

It is legitimate to believe that if Problem (3.1) can be closed exactly, *i.e.* without invoking *ad hoc* hypotheses, the Holy Grail for turbulence would then be uncovered. We show in the following that exact closure is actually possible.

Proposition 3.1. *Assuming Ω to be the 3D-torus, exact closure of* (3.1) *is then achievable.*

Proof. This result was pointed out by Germano in [20, 21]. Let $\varepsilon > 0$ be a cutoff scale. We consider the following filter (later referred to as the Helmholtz filter): for any given function $\mathbf{v}$, the filtered function $\overline{\mathbf{v}}$ is defined as the solution to the elliptic PDE:

$$\overline{\mathbf{v}} - \varepsilon^2 \nabla^2 \overline{\mathbf{v}} = \mathbf{v}, \tag{3.3}$$

that is, $\overline{\mathbf{v}} := (I - \varepsilon^2\nabla^2)^{-1}\mathbf{v}$. From a theorem by Agmon–Douglis–Nirenberg, it is shown that this filtering operator is continuous from $\mathbf{L}^q(\Omega)$, $1 \leq q < +\infty$, onto $\mathbf{W}^{2,q}(\Omega)$ (see *e.g.* [4, 68]). Furthermore, it can be shown that $\overline{(\cdot)}$ commutes with space and time derivatives. Hence, $\overline{(\cdot)}$ is intuitively an acceptable filter. Then, using the fact that

$$\mathbf{u} \otimes \mathbf{u} = (\overline{\mathbf{u}} - \varepsilon^2\nabla^2\overline{\mathbf{u}}) \otimes (\overline{\mathbf{u}} - \varepsilon^2\nabla^2\overline{\mathbf{u}}),$$

and since

$$\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} = \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \varepsilon^2\nabla^2\overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}},$$

it follows that

$$\begin{aligned}
\mathbb{T}_{ij} &= \overline{(\overline{u}_i - \varepsilon^2\nabla^2\overline{u}_i)(\overline{u}_j - \varepsilon^2\nabla^2\overline{u}_j)} - \overline{u}_i\overline{u}_j, \\
&= \overline{\overline{u}_i\overline{u}_j} - \varepsilon^2\overline{\overline{u}_j\nabla^2\overline{u}_i + \overline{u}_i\nabla^2\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j} - \overline{u}_i\overline{u}_j, \\
&= \varepsilon^2\overline{\nabla^2(\overline{u}_i\overline{u}_j) - \overline{u}_j\nabla^2\overline{u}_i - \overline{u}_i\nabla^2\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j} \\
&= 2\varepsilon^2\overline{\nabla\overline{u}_i \cdot \nabla\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j},
\end{aligned}$$

which is actually a closed-form of the subgrid-scale tensor. □

Remark 3.1. Another way to derive an exact closure involves defining the filter by means of a mollifier. Let ϕ be a bounded positive function in $\mathbb{R}^3$, fast decreasing at infinity, such that $\int_{\mathbb{R}^3}\phi\,\mathrm{d}\mathbf{x} = 1$, and such that its Fourier transform does not vanish. For instance, the Gaussian kernel $\phi(\mathbf{x}) = \pi^{-3/2}\exp(-|\mathbf{x}|^2)$ satisfies these hypotheses. Then for $\varepsilon > 0$, define $\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-3}\phi(\mathbf{x}/\varepsilon)$ and set

$$\overline{\mathbf{v}} := \phi_\varepsilon * \mathbf{v}. \tag{3.4}$$

Denoting by $\mathcal{F}$ the Fourier transform, we infer that

$$\forall\ \mathbf{u} \in \mathbf{L}^1(\Omega), \quad \mathbf{u} = \mathcal{F}^{-1}(\mathcal{F}(\overline{\mathbf{u}})/\mathcal{F}(\phi)).$$

Hence, in this case also, the subgrid-scale tensor can be expressed exactly in terms of the filtered field.

The preceding results can actually be generalized by observing that exact closure is achieved whenever the filtering operator induces an isomorphism. Indeed, filters (3.3) and (3.4) are isomorphisms. This is intuitively verified by the fact that they do not remove information from the field they are applied to; they simply rescale the spectrum of the field. For instance, given some $t > 0$, filter (3.3) induces isomorphisms between $L^\infty(0,t;\mathbf{H})$ and $L^\infty(0,t;\mathbf{H}\cap\mathbf{H}^2(\Omega))$ and between $L^2(0,t;\mathbf{V})$ and $L^2(0,t;\mathbf{V}\cap\mathbf{H}^3(\Omega))$. Hence, this filter induces an isomorphism between the set of weak solutions of (1.1) and that of (3.1). As a result, filtering and achieving exact closure is unlikely to improve the situation since, roughly speaking, given that the two sets of weak solutions are isomorphic, one should expect to use the same number of degrees of freedom for approximating the Navier–Stokes equations as for approximating the filtered equations.

Formally speaking, we are faced with the following paradox:

Filtering and achieving exact closure does not reduce the number of degrees of freedom.

Of course from a mathematical point of view paradoxes are impossible. Paradoxes are intimately related to human interpretations and human expectations. The present paradox we point out is that filtering the NS equation and closing them as accurately as possible, *i.e.* exactly, in the hope that the resulting problem will be simpler to solve is a paradoxical program.

In conclusion, the program which consists in filtering the Navier–Stokes equations in the hope of simplifying the daunting task of approximating them has some chance of success only if *inexact* closure is performed, *i.e.* information has to be removed.

3.3. Leray regularization

The first outstanding result which involves filtering is the proof of existence of weak solutions to the Navier–Stokes equation by Leray [47]. We assume here that Ω is the 3D-torus $(0, 2\pi)^3$. Denoting by $B(0, \varepsilon) \subset \mathbb{R}^3$ the ball of radius ε centered at 0, we consider a sequence of mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying:

$$\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad \operatorname{supp}(\phi_\varepsilon) \subset B(0,\varepsilon), \quad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1. \tag{3.5}$$

Defining the convolution product $\phi_\varepsilon * \mathbf{v}$ as:

$$\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y},$$

Leray suggested to regularize the Navier–Stokes equations as follows:

$$\begin{cases} \partial_t \mathbf{u} + (\phi_\varepsilon * \mathbf{u}) \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \phi_\varepsilon * \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \phi_\varepsilon * \mathbf{u}_0, \end{cases} \tag{3.6}$$

where, although the same variable names are used, it is understood that the solution of (3.6) is different from the solution of the Navier–Stokes equation (1.1). Leray proved the following theorem:

Theorem 3.1 (Leray [47])**.** *For all* $\mathbf{u}_0 \in \mathbf{H}$*,* $\mathbf{f} \in \mathbf{H}$*, and* $\varepsilon > 0$*,* (3.6) *has a* unique $\mathcal{C}^\infty$ *solution. This solution is also bounded in* $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ *and one subsequence converges weakly in* $L^2(0,T;\mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon \to 0$*.*

The striking result here is that the solution of (3.6) is unique. Hence, moderate filtering of the advection velocity (and if necessary, of the data $\mathbf{u}_0$ and $\mathbf{f}$) is

sufficient to guarantee uniqueness of $\mathcal{C}^\infty$ solutions; that is, it takes only a small amount of smoothing to ascertain that the energy cascade stops everywhere in the domain and for all times.

Rewriting the momentum equation of (3.6) as:

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \phi_\varepsilon * \mathbf{f} - (\phi_\varepsilon * \mathbf{u})\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} \tag{3.7}$$

and introducing the tensor $\mathbb{T}_L$ such that:

$$\nabla\cdot\mathbb{T}_L = (\phi_\varepsilon * \mathbf{u})\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} = \nabla\cdot(\mathbf{u}\otimes(\phi_\varepsilon * \mathbf{u}) - \mathbf{u}\otimes\mathbf{u}), \tag{3.8}$$

it is reasonable to think of the Leray regularization as a LES model. Equation (3.7) is indeed the same as the momentum equation of (3.1) except for the fact that the subgrid-scale tensor $\mathbb{T}$ is now approximated by $\mathbb{T}_L$. However, this interpretation is debatable, for the model is not frame invariant. We elaborate on this issue in the following section.

3.4. The Navier–Stokes-alpha model

Owing to the identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\mathbf{u}^2/2),$$

the Navier–Stokes problem can be rewritten in the form

$$\begin{cases} \partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\pi - \nu\nabla^2\mathbf{u} = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}|_\Gamma = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{3.9}$$

where $\pi = p + \frac{1}{2}\mathbf{u}^2$ is the total pressure. This form of the equations obviously raises the same uniqueness problem as the original form. This issue can nevertheless be resolved by resorting to the regularization technique proposed by Leray. Following the same strategy as in (3.6) and introducing the notation $\overline{\mathbf{u}} = \phi_\varepsilon * \mathbf{u}$, we are led to consider the following regularized problem

$$\begin{cases} \partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\overline{\mathbf{u}} + \nabla\pi - \nu\nabla^2\mathbf{u} = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{3.10}$$

Using the identities,

$$\begin{aligned} (\nabla\times\mathbf{u})\times\overline{\mathbf{u}} &= \overline{\mathbf{u}}\cdot\nabla\mathbf{u} - (\nabla\mathbf{u})^T\overline{\mathbf{u}}, \\ \nabla(\mathbf{u}\cdot\overline{\mathbf{u}}) &= (\nabla\mathbf{u})^T\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\mathbf{u}, \end{aligned} \tag{3.11}$$

we can recast system (3.10) into the following equivalent form,

$$\begin{cases} \partial_t \mathbf{u} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u} + (\nabla \overline{\mathbf{u}})^T \cdot \mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla \pi' = \mathbf{f}, \\ \nabla \cdot \overline{\mathbf{u}} = 0, \\ \mathbf{u},\ \overline{\mathbf{u}} \text{ are periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{3.12}$$

where $\pi' = \pi - \mathbf{u} \cdot \overline{\mathbf{u}}$.

Upon choosing the Helmholtz filter, introduced in the proof of Proposition 3.1 and defined as

$$\overline{\mathbf{v}} := (I - \varepsilon^2 \nabla^2)^{-1} \mathbf{v}, \tag{3.13}$$

we recognize the model thoroughly analyzed in Chen *et al.* [9] and Foias *et al.* [15, 16]. Regarding the notational choice $\alpha = \varepsilon$, the authors refer to this particular regularization as the "Navier–Stokes-alpha model". Once again, regularization yields uniqueness and the expected regularity properties as stated in the following theorem:

Theorem 3.2 (Foias, Holm and Titi [15, 16])**.** *Assume* $\mathbf{f} \in \mathbf{H}$*,* $\mathbf{u}_0 \in \mathbf{V}$*. Problem* (3.12) *with the Helmholtz filter* (3.13) *has a* unique *regular solution* $\mathbf{u}$*. The solution* $\overline{\mathbf{u}}$ *is bounded in* $L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V})$ *and one subsequence converges weakly in* $L^2_{loc}(0, +\infty; \mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon \to 0$*.*

Numerical simulations reported in [15] show that the energy spectrum of the solution of (3.12) follows the $k^{-5/3}$ law for $k \lesssim 1/\varepsilon$ and rolls off to k^{-3} for $k \gtrsim 1/\varepsilon$. Hence, below the scale ε the regularization has replaced the $k^{-5/3}$ tail of the spectrum, which is difficult to approximate numerically, by a more gentle k^{-3} tail.

At this point of the analysis, it seems natural to ask which of $\mathbf{u}$ or $\overline{\mathbf{u}}$ is the most "physically relevant" quantity. Here, by "physically relevant", we mean that the quantity is frame-invariant. By a change of frame of reference in (3.12) with respect to $\mathbf{u}$, it can be shown that $\mathbf{u}$ is not physically relevant in this sense (see [28]). In order to analyze the frame-dependence of the filtered velocity, we first rewrite (3.12) in terms of $\overline{\mathbf{u}}$. Introducing the strain rate tensor $\mathbf{D} = \frac{1}{2}(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T)$, the vorticity tensor $\mathbf{\Omega} = \frac{1}{2}(\nabla \overline{\mathbf{u}} - (\nabla \overline{\mathbf{u}})^T)$, and the Jaumann derivative [62]

$$\mathring{\mathbf{D}} = \partial_t \mathbf{D} + \overline{\mathbf{u}} \cdot \nabla \mathbf{D} + \mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D}$$

we show that (3.12) is equivalent to the following system:

$$\begin{cases} \partial_t \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} = \nabla \cdot \mathbf{T}, \\ \qquad \text{with } \mathbf{T} = -p\mathbf{I} + 2\nu(1 - \varepsilon^2 \nabla^2)\mathbf{D} + 2\varepsilon^2 \mathring{\mathbf{D}}, \\ \nabla \cdot \overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}} \text{ is periodic}, \\ \overline{\mathbf{u}}|_{t=0} = \overline{\mathbf{u}}_0 \end{cases} \tag{3.14}$$

We recognize here the constitutive law of a rate-dependent incompressible homogeneous fluid of second grade in which the dissipation is slightly modified by composition with the Helmholtz operator (see [15] for more details). This set of equations is frame-invariant; hence, according to our definition, $\overline{\mathbf{u}}$ is the quantity that is "physically relevant."

In conclusion, the regularization procedure using the Helmholtz filter (3.13), as performed in (3.10), is strictly equivalent to LES where the subgrid tensor is now modeled as follows

$$\mathbb{T} = 2\nu\varepsilon^2\nabla^2\mathbf{D} - 2\varepsilon^2\mathring{\mathbf{D}}. \tag{3.15}$$

The first term amounts to adding some hyper-viscosity (*i.e.* a bilaplacian) whereas the second one introduces dispersion effects.

Finally, if in the Leray-regularized equations (3.6) the regularization is performed by using the Helmholtz filter, we observe that the equations can also be rewritten in terms of the filtered velocity $\overline{\mathbf{u}}$ only. In this case, we obtain a set of equations for $\overline{\mathbf{u}}$ that are almost frame-invariant but for the term $\varepsilon^2(\nabla\overline{\mathbf{u}})^T\nabla^2\overline{\mathbf{u}}$. Quite surprisingly, if this $\mathcal{O}(\varepsilon^2)$ term is simply neglected from the momentum equation, we recover the Navier–Stokes-alpha model (3.12) (see [28]). Hence, the Navier–Stokes-alpha model can be viewed as a frame-invariant version of the Leray regularization. The progress made from the ground-breaking work of Leray in 1934 is deceivingly small.

3.5. The local energy equilibrium

Another intriguing problem related to the uniqueness question is that of local energy balance. Though it is quite simple to show that weak solutions of the Navier–Stokes equations satisfy a global energy balance (2.3), it has not yet been possible to prove that a local energy balance holds. In this respect, the following result is proved in Duchon and Robert [13, 63]:

Proposition 3.2. *Let $\mathbf{u} \in L^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$ be a weak solution of the Navier–Stokes equation in the 3D-torus with no source term. Let $\phi_\varepsilon(\mathbf{x}) = (1/\varepsilon^3)\phi(\mathbf{x}/\varepsilon)$ be a mollifying sequence (i.e., ϕ is even, $\mathcal{C}^\infty$ with compact support in $\mathbb{R}^3$, non-negative and normalized). Set $\mathcal{D}_\varepsilon(\mathbf{u})(\mathbf{x}) = \frac{1}{4}\int \nabla\phi_\varepsilon(\mathbf{y})\cdot\delta\mathbf{u}(\delta\mathbf{u})^2\,\mathrm{d}y$, where $\delta\mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x})$. Then $\mathcal{D}_\varepsilon(\mathbf{u})$ converges in $\mathcal{D}'(]0,T[\times\Omega)$ to a distribution $\mathcal{D}(\mathbf{u})$ that does not depend on ϕ and such that the following energy balance holds:*

$$\partial_t(\tfrac{1}{2}\mathbf{u}^2) + \nabla\cdot(\mathbf{u}(\tfrac{1}{2}\mathbf{u}^2 + p)) - \nu\nabla^2\tfrac{1}{2}\mathbf{u}^2 + \nu(\nabla\mathbf{u})^2 + \mathcal{D}(\mathbf{u}) = 0.$$

For each solution $\mathbf{u}$, "$\mathcal{D}(\mathbf{u})$ measures a possible dissipation or production of energy caused by lack of smoothness in the velocity field." This quantity is zero only if $\mathbf{u}$ is smooth enough. For instance, it is possible to prove that $\mathcal{D}(\mathbf{u}) = 0$ if $\int|\mathbf{u}(t,\mathbf{x}+\mathbf{y}) - \mathbf{u}(t,\mathbf{x})|^3\,\mathrm{d}y \le c(t)|\mathbf{y}|\sigma(|\mathbf{y}|)$ where $\int_0^T c(t)\,\mathrm{d}t < +\infty$ and σ is continuous in 0 with $\sigma(0) = 0$ (see [13]).

As pointed out by Duchon and Robert [13, 63], it is remarkable that every solution of the Navier–Stokes equations obtained as a limit of (a subsequence of) solutions $\mathbf{u}_\varepsilon$ of the regularized equations introduced by Leray (3.6) is "dissipative" in the sense that $\mathcal{D}(\mathbf{u}) \geq 0$; that is, these solutions of the Navier–Stokes equations satisfy

$$\partial_t(\tfrac{1}{2}\mathbf{u}^2) + \nabla\cdot(\mathbf{u}(\tfrac{1}{2}\mathbf{u}^2 + p)) - \nu\nabla^2\tfrac{1}{2}\mathbf{u}^2 + \nu(\nabla\mathbf{u})^2 \leq 0. \tag{3.16}$$

When $\mathbf{u}$ is obtained as a limit of a finite dimensional Galerkin approximation, the sign of $\mathcal{D}(\mathbf{u})$ is unknown; that is, the lack of smoothness of Galerkin solutions might lead to local energy creation. This counter-intuitive result sheds doubt on the physical relevance of Galerkin solutions, for one would expect lack of smoothness to always dissipate energy. In other words, assuming we compute an approximate solution of the Navier–Stokes equations using the Galerkin technique on a finite element mesh and we make the meshsize go to zero, then we are not guaranteed that the limit solution satisfy (3.16), whereas if we regularize the advection term *à la* Leray, make the meshsize go to zero, and make the regularization parameter go to zero afterward, then the limit solution necessarily satisfies (3.16).

It is also remarkable that the notion of "dissipative solution" introduced by Duchon and Robert coincides with the notion of "suitable weak solutions" introduced by Caffarelli *et al.* [6] (based on the work of Scheffer [67]) for which the best partial regularity theorem to date has been proved [6].

The arguments developed above can be summarized as follows.

(i) Limit of Galerkin approximations of the LES Navier–Stokes equations using the Leray regularization or the NS-α model are "dissipative" (or "suitable weak solutions") whereas limits of Galerkin approximations may not be so. The first limit has to be understood in the sense $\lim_{\varepsilon\to 0}\lim_{h\to 0}$, h being the discretization parameter, whereas the second one is $\lim_{h\to 0}$.

(ii) Limits of regularized solutions are possibly more regular than limits of Galerkin approximations.

Hence, limits of LES solutions may be physically more relevant than limits of Galerkin approximations, thus justifying LES strategies *à la* Leray.

4. The p-Laplacian models

In this section, LES models based on nonlinear viscosity are analyzed. We review in particular the Smagorinsky model and unravel similarities with the models introduced by Ladyženskaja and Kaniel and with the p-Laplacian regularization techniques. The major conclusions from the analysis presented in this section are summarized as follows:

(i) This class of models for LES, and in particular the Smagorinsky model, are justified from a mathematical point of view as they regularize the Navier–Stokes equations; *i.e.* they yield a well-posed problem.

(ii) We also show that the p-Laplacian regularization presents interesting numerical properties that make possible the derivation of L^∞-error estimates.

4.1. The Smagorinsky model

Possibly one of the most popular models for Large Eddy Simulations is that proposed by Smagorinsky [69]. The model consists in adding to the stress tensor an additional nonlinear viscous term that depends on a small length scale ε to be fixed in some *ad hoc* way. Denoting the deformation tensor by $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, the additional tensor is written in the form

$$\varepsilon^2|\mathbf{D}|\mathbf{D} \tag{4.1}$$

Upon introducing the notation $\mathbf{T}(\nabla\mathbf{u}) = |\mathbf{D}|\mathbf{D}$, the perturbed Navier–Stokes equations are

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot(\nu\nabla\mathbf{u} + \varepsilon^2\mathbf{T}(\nabla\mathbf{u})) = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}|_\Gamma = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{4.2}$$

In the literature, another way of presenting the method consists in stating that (4.2) is the filtered Navier–Stokes equations where the subgrid tensor is modeled by $\mathbb{T} = -\varepsilon^2\mathbf{T}(\nabla\mathbf{u})$ (see *e.g.* Lilly [52] or Muschinsky [59]). We think that invoking filters here is questionable, or at least debatable, on three accounts. First, introducing a non-Newtonian effect to model turbulence is perfectly justified phenomenologically without invoking filtering as it has been remarked by Rivlin [61]: "It has been reported [...] that when a Newtonian fluid flows down a straight pipe of non-circular cross-section, under conditions for which the fluid has become fully turbulent, the [mean] flow is no longer rectilinear, but a secondary flow exists in the cross-sectional planes of a type similar to that [that can be calculated for some non-Newtonian fluids]. This fact suggests that the turbulent Newtonian liquid may, for certain purposes, be regarded as a non-Newtonian fluid." Second, it is also perfectly justified from a numerical point of view. Since the pioneering work of von Neumann and Richtmyer [73], it is well known that introducing nonlinear viscosity helps selecting physically relevant solutions of nonlinear conservation laws (*i.e.* entropic solutions) without spoiling too much the accuracy of the numerical method used. Third, contrary to the Leray regularization and the NS–alpha model where filtering is at the root of the models, the mathematical analysis of the well-posedness of the Smagorinsky model (4.2) (see §4.2.) does not make use of the notion of filter.

We now come to a striking result due to Ladyženskaja [43, 42] that sheds an original light on Smagorinsky's model, namely that the model actually yields a well-posed problem.

4.2. The Ladyženskaja model

Recalling that the Navier–Stokes equations are based on Newton's linear hypothesis, Ladyženskaja and Kaniel proposed to modify the incompressible Navier–Stokes equations to take into account possible large velocity gradients, [43, 42, 37].

Ladyženskaja introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla\mathbf{u})$, $1 \leq i,j \leq 3$ satisfying the following conditions:

L1. $\mathbf{T}$ is continuous and there exists $\mu \geq \frac{1}{4}$ such that

$$\forall\ \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad |\mathbf{T}(\boldsymbol{\xi})| \leq c(1+|\boldsymbol{\xi}|^{2\mu})|\boldsymbol{\xi}|. \tag{4.3}$$

L2. $\mathbf{T}$ satisfies the coercivity property:

$$\forall\ \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad \mathbf{T}(\boldsymbol{\xi}) : \boldsymbol{\xi} \geq c|\boldsymbol{\xi}|^2(1+c'|\boldsymbol{\xi}|^{2\mu}). \tag{4.4}$$

L3. $\mathbf{T}$ possesses the following monotonicity property: there exists a constant $c > 0$ such that for all solenoidal fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ in $\mathbf{W}^{1,2+2\mu}(\Omega)$ either coinciding on the boundary Γ or being periodic,

$$\int_\Omega (\mathbf{T}(\nabla\boldsymbol{\xi}) - T(\nabla\boldsymbol{\eta})) : (\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}) \geq c\int_\Omega |\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}|^2. \tag{4.5}$$

These conditions are actually satisfied in the case where

$$\mathbf{T}(\boldsymbol{\xi}) = \beta(|\boldsymbol{\xi}|^2)\boldsymbol{\xi} \tag{4.6}$$

provided the viscosity function $\beta(\tau)$ is a positive monotonically-increasing function of $\tau \geq 0$ and for large values of τ the following inequality holds

$$c\tau^\mu \leq \beta(\tau) \leq c'\tau^\mu,$$

with $\mu \geq \frac{1}{4}$ and c, c' are some strictly positive constants. Smagorinsky's model obviously falls into the admissible category with $\beta(\tau) = \tau^{1/2}$.

Introducing now a (possibly small) positive constant $\varepsilon > 0$, the modified Navier–Stokes equations take the form

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot(\nu\nabla\mathbf{u} + \varepsilon\mathbf{T}(\nabla\mathbf{u})) = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}|_\Gamma = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{4.7}$$

The striking result from [43, 42] (see [37] for a similar result where monotonicity is also assumed) is the following theorem

Theorem 4.1. *Provided conditions* L1, L2, *and* L3 *are satisfied,* $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{f} \in L^2(]0,+\infty[;\mathbf{L}^2(\Omega))$, *then* (4.7) *has a* unique *weak solution in*

$$L^{2+2\mu}(]0,t[;\mathbf{W}^{1,2+2\mu}(\Omega)\cap\mathbf{V}) \cap C^0([0,t];\mathbf{H}) \quad \textit{for all } t > 0.$$

Note that uniqueness in ensured for times t possibly arbitrarily large. This result states that a small appropriate amount of nonlinear viscosity is actually sufficient to ascertain that the energy cascade stops, which automatically translates into uniqueness of solutions for arbitrary times. Moreover, it is possible to prove the following theorem.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, the sequence of solutions to* (4.7) *has subsequences that converge weakly in* $L^2(0,T;\mathbf{V})$ *to weak Navier–Stokes solutions as* $\varepsilon \to 0$ *and these solutions are dissipative in the sense of Duchon and Robert* (i.e. *satisfy* (3.16)).

In conclusion, perturbing the Navier–Stokes equations with a term like Smagorinsky's model solves the uniqueness question and ensures that the limit solution are dissipative, thus giving a strong mathematical support for this type of LES models.

4.3. On tuning the Smagorinsky constant and numerical issues

When one comes to the question of building a numerical approximation of the solution to (4.2), one immediately stumbles on the problem of choosing the small length scale ε in the definition of the Smagorinsky tensor (4.1). To emphasize that ε is a length scale, let us make henceforth the following change of notation $\ell_{\text{LES}} = \varepsilon$.

In most of current literature, the authors set $\ell_{\text{LES}} = c_s h$ where h is the mesh size of the numerical method used, and the constant c_s is tuned so that the model reproduces the $k^{-5/3}$ cascade when simulating isotropic turbulence in the 3D-torus. We refer to Lilly [52, 53] and Germano *et al.* [22] for landmark papers proposing tentative evaluations of these parameters. The literature dedicated to this problem is voluminous and somewhat controversial, and we do not feel qualified to discuss the many physical arguments invoked; however, the mathematical arguments advanced below suggest that the standard choice $\ell_{\text{LES}}/h = c_s \approx 0.18$ is debatable.

4.3.1. The mathematical viewpoint

Our first argument is that, generally speaking, amalgamating in some *ad hoc* way a mathematical model with a computational one constitutes a serious crime. To better appreciate this point of view, we refer to the mathematical analysis of (4.2) (resp. (4.7)). In general, existence of solutions is proven using the Galerkin technique or one of its variants (see *e.g.* Lions [54]). In other words, one considers a sequence of approximations $(\mathbf{u}_h)_{h>0}$ (for example, numerical approximations) defined on a sequence of finite dimensional spaces indexed by h. Then, upon using *a priori* estimates uniform in h along with compacity results, one can prove

that the sequence $(\mathbf{u}_h)_{h>0}$ converges to the unique solution of (4.2) (resp. (4.7)) in some appropriate topology. It follows that the solution to (4.2) (resp. (4.7)) should be understood as the limit of numerical approximations $\mathbf{u}_h$ as $h \to 0$ while $\ell_{\rm LES}$ is being held fixed. In other words, if we set $c_s = \ell_{\rm LES}/h$, we obtain that c_s must grow as h goes to zero, which is clearly in contradiction with the conventional notion of c_s being a constant.

4.3.2. The energy viewpoint

This point of view has actually been defended by Muschinsky [59] who advanced heuristic arguments that we summarize below. Upon assuming that the subgrid tensor is given by $2\nu_{\rm LES}\mathbf{D}$ where $\nu_{\rm LES} = \ell_{\rm LES}(2\mathbf{D}:\mathbf{D})^{1/2}$, the dissipation is then given by

$$\epsilon = \langle 2\nu_{\rm LES}\mathbf{D} : \nabla\mathbf{u}\rangle = \ell_{\rm LES}2^{3/2}\langle(\mathbf{D}:\mathbf{D})^{3/2}\rangle \tag{4.8}$$

where $\langle\cdot\rangle$ is some appropriate average in time and space. Now, the conventional wisdom consists is assuming the turbulence in the flow to be homogeneous and isotropic so that

$$\langle \mathbf{D}:\mathbf{D}\rangle = \int_0^{+\infty} E(k)k^2 \mathrm{d}k, \tag{4.9}$$

where $E(k)$ is the distribution of kinetic energy. If there was no dissipation (*i.e.*, $\nu = \nu_{\rm LES} = 0$), the Kolmogorov cascade would prevail at every scale in the flow and one would have $E(k) = C_K\epsilon^{2/3}k^{-5/3}$ (henceforth we use the value $C_K = 1.5$ which is reasonably representative of experimental data). But, in the case of numerical simulations on finite grids with a viscous term *à la* Smagorinsky, it is unreasonable to assume that these dissipative effects are not felt by the kinetic energy distribution. Hence, Muschinsky proposes to consider

$$E(k) = C_{\rm LES}\epsilon^{2/3}k^{-5/3}f(k\ell_{\rm LES}, \ell_{\rm LES}/h), \tag{4.10}$$

where f is some *ad hoc*, hopefully universal, dissipation function and $C_{\rm LES}$ should be as close as possible to the Kolmogorov constant C_K. Upon inserting (4.10) into (4.9) and using the result in (4.8), one obtains

$$C_{\rm LES} = \frac{1}{2\int_0^{+\infty} x^{1/3}f(x, \ell_{\rm LES}/h)\mathrm{d}x}. \tag{4.11}$$

In this framework, the first attempt by Lilly [52] to compute the Smagorinsky constant consists in assuming that the dissipation function does not depend on the ratio $\ell_{\rm LES}/h$ and can be approximated as follows

$$f_L(x) = \begin{cases} 1 & \text{if } k \le \pi/h \\ 0 & \text{if } k > \pi/h. \end{cases}$$

Muschinsky argues that this type of function is not realistic since it consists in assuming that above scale π/h no dissipation is felt, whereas below this scale all motions are frozen. As an alternative to this crude point of view, the author

proposes to consider the following semi-empirical dissipation function introduced by Heisenberg [29]

$$f_H(x) = \left[1 + \left(\frac{3C_K}{2}\right)^3 x^4\right]^{-4/3}$$

and to replace the $+\infty$ bound in (4.11) by $\pi\ell_{\rm LES}/h$. The graphs of f_L and f_H are shown in Figure 1.

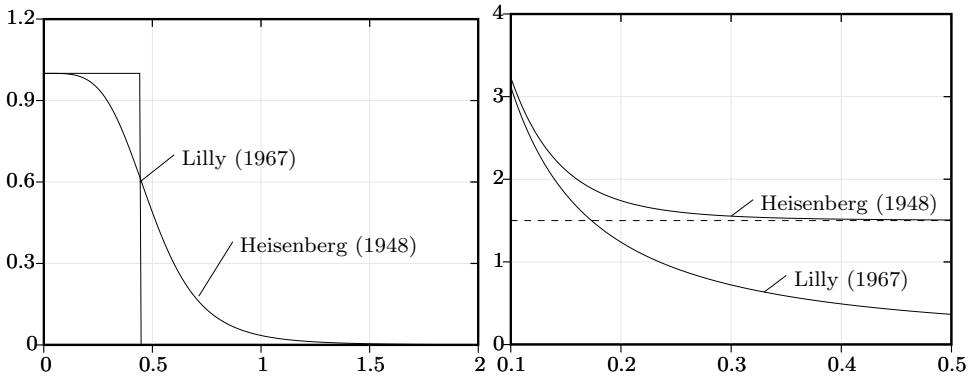

FIG. 1. Models for the dimensionless dissipation spectrum for locally homogeneous and isotropic turbulence generated by a Smagorinsky-type LES. Left: graph of the dissipation functions f_L and f_H according to the models by Lilly [52] and Heisenberg [29], respectively. Right: graph of $C_{\rm LES}$ vs. c_s using the dissipation functions f_L and f_H.

Using either f_L or f_H in (4.11), and introducing the notation $c_s = \ell_{\rm LES}/h$, one immediately obtains

$$C_{\rm LES,L} = \frac{2}{3}(\pi c_s)^{-4/3} \tag{4.12}$$

$$C_{\rm LES,H} = C_K\left[1 + \left(\frac{2}{3C_K}\right)^3 (\pi c_s)^4\right]^{1/3}, \tag{4.13}$$

where c_s is still a free parameter. The plots of $C_{\rm LES,L}$ and $C_{\rm LES,H}$ versus c_s are shown in Figure 1. Now the traditional wisdom consists in choosing c_s such that $C_{\rm LES}$ is as close as possible to the Kolmogorov constant $C_K = 1.5$. It is clear that by adopting the viewpoint of Lilly [52], there is only one possible choice, $c_s = (2/3C_K)^{3/4}/\pi \approx 0.18$. Henceforth let us denote $c_0 = (2/3C_K)^{3/4}/\pi$. For $C_{\rm LES,H}$ to be close to C_K, the correct "mathematical" choice is $c_s = +\infty$, and for practical purposes $2 \times c_0 \approx 0.36 \lesssim c_s$ seems adequate. Of course, in practice, admissible values of c_s are bounded from above by the constraint $\ell_{\rm LES} = c_s h \ll L$. Moreover, Muschinsky argues that c_s must be small enough in order for the turbulent Reynolds number $R_{e_t} = (L/c_s h)^{4/3}$ to be large enough, so that an

inertial cascade is sustained. As a result a value of c_s twice as large as (or possibly larger than) the traditional value proposed by Lilly is recommended.

4.3.3. Computational evidence

We finish this section by noting that this viewpoint seems to have been gaining momentum within the last few years. In particular, numerical simulations of forced and freely decaying turbulence in the 3D-torus by Magnient *et al.* [56] confirm energy distributions like (4.10) where $f(x)$ is close to the Heisenberg dissipation function. Moreover, the authors show that by choosing $\ell_{\mathrm{LES,H}} = c_0 h$, with $c_0 = 0.18$, the energy distribution depends widely on the numerical method used to solve the equations (the authors have tested second-order, fourth-order and sixth-order compact finite differences schemes) and in these circumstances obtaining a $k^{-5/3}$ spectrum is a matter of selecting the "right" numerical scheme (not necessarily the most accurate one). Their numerical experiments [56, Fig. 11] clearly show that by using $\ell_{\mathrm{LES}} \geq c_0 4h$, the energy spectrum is almost independent of the numerical scheme, a property that is generally looked for in numerical analysis. Similar conclusions have been drawn by Ghosal [23]. By referring to ℓ_{LES} as the LES filter-width and to $c_0 h$ as the characteristic size of the computational grid, the author writes: "In order that the smallest resolved scales be representable on the grid, it is required that $c_0 h \leq \ell_{\mathrm{LES}}$ [...] As a matter of fact, this distinction between $c_0 h$ and ℓ_{LES} is often ignored [...] However, if one expects to adequately resolve all scales up to ℓ_{LES} it is natural to require that $c_0 h$ be several times smaller than ℓ_{LES}. Thus, we are led to consider an LES with a filter-width ℓ_{LES} performed on a numerical grid of spacing $c_0 h < \ell_{\mathrm{LES}}$." The author finally recommends to take at least $c_0 2h \leq \ell_{\mathrm{LES}}$. A ratio 2 or 4 may not seem to be significant until one recalls that c_s comes into play in (4.2) by its square. If one choses $c_s/c_0 = 3$, then it is clear that c_s^2 is larger than c_0^2 by one order of magnitude, *i.e.* $c_s^2 \gg c_0^2$.

Finally, although Smagorinsky's turbulence model is acclaimed for its remarkable ability to reproduce the $k^{-5/3}$ energy spectrum, the theoretical results of §4.2. show that this model is nothing but a regularization technique among many others; in fact, the various models proposed by Ladyženskaja [43, 42] and Kaniel [37] should achieve similar results.

4.4. The p-Laplacian

The p-Laplacian operator is a simple version of both Smagorinsky's and Ladyženskaja's models. This operator is well-known to mathematicians for being a prototype for monotone operators, see *e.g.* Lions [54] or Showalter [68]. We show in this section how this operator can be used for approximating advection dominated advection-diffusion equations, hence giving a second mathematical interpretation of Smagorinsky's model.

Assuming $\Omega \subset \mathbb{R}^d$ and $p \geq 2$, the p-Laplacian is the operator defined as follows:

$$T_p : W_0^{1,p}(\Omega) \ni u \longmapsto T_p(u) = -\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) \in W_0^{1,p}(\Omega)'.$$

It is clearly bounded in $W_0^{1,p}(\Omega)$ and satisfies the following monotonicity property

$$\exists \alpha > 0, \forall\ u, v \in W_0^{1,p}(\Omega), \qquad \langle T_p(u) - T_p(v), u - v\rangle \geq \alpha \|\nabla(u-v)\|_{1,p}^p,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing (the reader not familiar with this notion can view it as the L^2-scalar product). Note that when $p = 2$, T_p is nothing else but the Laplacian. We also observe that Smagorinsky's model corresponds to the p-Laplacian with $p = 3$.

This operator has interesting properties owing to the following standard Sobolev embedding: there exists $c > 0$ such that for all $u \in W^{1,p}(\Omega)$

$$\|u\|_{L^q(\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)} \begin{cases} \frac{1}{q} = \frac{1}{p} - \frac{1}{d} & \text{if } 1 \leq p < d \\ p \leq q < \infty & \text{if } p = d \\ q = +\infty & \text{if } p > d \end{cases} \tag{4.14}$$

where d is the space dimension (see *e.g.* [4, 68]). Hence the monotonicity property yields *a priori* estimates in the norm of $W^{1,p}(\Omega)$, which not only controls the standard $H^1(\Omega)$-norm, but also controls the norm of $L^q(\Omega)$. For instance, for $p > 3$ this term allows for a control in the $L^\infty(\Omega)$ norm in three dimensions.

To illustrate these ideas, we consider an advection diffusion equation dominated by advection:

$$\begin{cases} -\varepsilon\nabla^2 u + \boldsymbol{\beta}\cdot\nabla u = f, \\ u|_\Gamma = 0, \end{cases} \tag{4.15}$$

where $\varepsilon > 0$. For the sake of simplicity, we assume that $\boldsymbol{\beta}$ is a smooth solenoidal vector field with zero normal trace on Γ, *i.e.* $\boldsymbol{\beta} \in \mathbf{H}$, and ε is some positive real number. It is well-known that approximating this type of equation is a nontrivial task when the ratio $h\|\boldsymbol{\beta}\|_{0,\infty}/\varepsilon$ is large, h being the typical meshsize. Standard Galerkin (*i.e.* centered) approximation yields spurious node-to-node oscillations, the heuristic reason for this being that the grid is not fine enough for the viscous effects to dampen gradients. This problem is to be put in parallel with the difficulties in approximating fluid flows at high Reynolds numbers. In the same spirit as that of the Smagorinsky model, one may try to slightly modify the problem by adding some nonlinear viscosity.

Let $X_h \subset H_0^1(\Omega)$ be a finite dimensional space having standard interpolation properties (for instance a continuous $\mathbb{P}_k$ finite element space), *i.e.* there are $c > 0$ and $k > 0$ such that for all $v \in W^{1,p}(\Omega)$

$$\inf_{v_h \in X_h} \left(\|v - v_h\|_{0,p} + h\|v - v_h\|_{1,p}\right) \leq ch^{k+1}\|v\|_{k+1,p}. \tag{4.16}$$

Let us set

$$a(u,v) = (\nabla u, \nabla v), \qquad b(u,v) = (\boldsymbol{\beta}\cdot\nabla u, v), \tag{4.17}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. We consider the following approximate problem

$$\begin{cases} \text{Find } u_h \text{ in } X_h \text{ s.t.} \\ \varepsilon a(u_h, v_h) + b(u_h, v_h) + h^\sigma (T_p(u_h), v_h) = (f, v_h), \quad \forall\ v_h \in X_h, \end{cases} \tag{4.18}$$

where σ and $p \geq 2$ are yet to be determined. It is clear that we perturb the original problem by a term which is $\mathcal{O}(h^\sigma)$. Hence, to preserve optimal convergence estimates on gradients one should choose σ such that $\sigma \geq k$.

Theorem 4.3. *Under above hypotheses and provided u is smooth enough, we have the following error estimate*

$$\|u - u_h\|_{1,p} \leq (h^{\frac{1}{p-1}} + h^{\frac{k+1-\sigma}{p-1}}) c(u),$$

where $c(u) = c \max(\|u\|_{1,p}, h^k \|u\|_{k+1,p}, \|u\|_{k+1,p'}^{\frac{1}{p-1}})$.

The interesting aspect of such an estimate is that it is uniform with respect to ε, though ε is implicitly accounted for in $c(u)$. Note also that it is necessary to impose $k+1 > \sigma$ in order to achieve convergence. As a result, consistency and $W^{1,p}$-convergence are guaranteed if

$$k \leq \sigma < k+1.$$

Of course, for the case $\sigma = 1$ and $p = 2$, which corresponds to the crude first order linear viscosity, we obtain $\|u - u_h\|_{1,2} \leq ch$. But a more interesting situation arises if $p > d$, because convergence in the L^∞-norm is then guaranteed due to the Sobolev inequality (4.14). In this case, convergence occurs without unbounded spurious oscillations. Note finally that in three dimensions and for a second order method, *i.e.* $k = 1$, the limit case to obtain L^∞-convergence is $p = 3$ and $\sigma = k+1 = 2$. That is, the limit case is formally equivalent to the Smagorinsky model.

5. Spectral viscosity methods

In this section we review the so-called spectral eddy-viscosity methods. These LES techniques, initially introduced by Kraichnan [41], are frequently used in conjunction with spectral approximation methods. A voluminous literature has been dedicated to these methods and the approach on which they are based relies on sophisticated physical arguments that we do not assess. Our primary goal here is to identify mathematical arguments that can justify these techniques.

This section is organized as follows. We first recall the approach proposed by Kraichnan [41]. We then suggest a striking similarity between this approach and the method developed by Tadmor [72] for solving nonlinear scalar conservation equations. We proceed by comparing these two methods and by underlining some of their differences. Since the two methods are very similar in the way they are

numerically implemented, we also inquire whether the theory of the spectral viscosity method can be viewed as a mathematical justification for the eddy-viscosity method.

The main conclusions of this section are as follows:

(i) These methods are essentially numerical, and are linear or quasi-linear regularizations.

(ii) Eddy-viscosity methods do not achieve spectral accuracy, contrary to spectral viscosity methods.

(iii) The addition of a linear viscosity to nonlinear scalar conservation laws guarantees convergence of the approximate solution to the unique entropy solution. Unfortunately, applying this concept to the Navier–Stokes equations does not solve the uniqueness question. Hence, contrary to the filtering techniques and the nonlinear viscosity regularizations, this class of methods, in their original forms, do not regularize the Navier–Stokes equations.

(iv) We finally propose a minor, seemingly new, modification of the eddy-viscosity that solves the uniqueness question.

5.1. Kraichnan's eddy-viscosity

A Fourier approximation of the Navier–Stokes equations in the 3D-torus is built as follows. Given a cut-off wavenumber, $k_c > 1$, we denote by $\mathbf{P}_{k_c}$ the $\mathbf{L}^2$-projection onto $\mathbf{H}$ of vector-valued k_c-trigonometric polynomials:

$$\mathbf{P}_{k_c}(\mathbf{v}) = \hat{\mathbf{v}}_0 + \sum_{\substack{\mathbf{k}\in\mathbb{Z}^3 \\ |\mathbf{k}|_\infty \le k_c, \mathbf{k}\neq 0}} \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right) \hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Note that the operator $\mathbf{P}_{k_c}$ commutes with derivatives. The approximate solution $\mathbf{u}_{k_c}(\mathbf{x},t) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, satisfies

$$\begin{cases} \partial_t \mathbf{u}_{k_c} - \nu \mathbf{P}_{k_c} \nabla^2 \mathbf{u}_{k_c} + \mathbf{P}_{k_c}(\mathbf{u}_{k_c} \cdot \nabla \mathbf{u}_{k_c}) = \mathbf{P}_{k_c} \mathbf{f}, \\ \mathbf{u}_{k_c}|_{t=0} = \mathbf{P}_{k_c} \mathbf{u}_0. \end{cases} \tag{5.1}$$

Of course, if k_c is not large enough, *i.e.* if the Kolmogorov scale $\lambda \sim L R_e^{-3/4}$ is much smaller than the grid size $2\pi k_c^{-1}$, then energy accumulates at the cut-off scale. This translates in practice into spurious node-to-node oscillations in the approximate solution. The main purpose of LES is to avoid this phenomenon by adding extra terms to the Navier–Stokes equations so that the energy dissipates through the Kolmogorov cascade at the right rate. Assuming that the cut-off is large enough so that k_c is in the inertial range of the cascade, then Kraichnan [41] drew upon *ad hoc* statistical hypotheses that an eddy-viscosity should be added

to (5.1) and he proposed to modify the approximation scheme as follows:

$$\begin{cases} \partial_t \mathbf{u}_{k_c} - \mathbf{P}_{k_c}\nabla^2(\nu \mathbf{u}_{k_c} + \nu_t(k_c) Q_{k_c} * \mathbf{u}_{k_c}) + \mathbf{P}_{k_c}(\mathbf{u}_{k_c}\cdot\nabla \mathbf{u}_{k_c}) = \mathbf{P}_{k_c}\mathbf{f}, \\ \mathbf{u}_{k_c}|_{t=0} = \mathbf{P}_{k_c}\mathbf{u}_0, \end{cases} \tag{5.2}$$

with the vanishing viscosity $\nu_t(k_c)$ being such that

$$\nu_t(k_c) = E(k_c)^{1/2} k_c^{-1/2}, \tag{5.3}$$

where $E(k_c) = \frac{L}{2\pi}\frac{1}{2}\sum_{|\mathbf{k}|_\infty = k_c} |\hat{\mathbf{u}}_\mathbf{k}|^2$ is the kinetic energy at the cut-off scale[1] and the viscosity kernel assumes the form

$$Q_{k_c}(\mathbf{x}) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{\nu}_t(|\mathbf{k}|_\infty) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.4}$$

where $\hat{\nu}_t$ is a non-dimensional function which is constant for $|\mathbf{k}|_\infty/k_c \lesssim 0.3$, but increases for higher values of the ratio $|\mathbf{k}|_\infty/k_c$ so that the graph of $\hat{\nu}_t$ makes an upward cusp in the vicinity of k_c (see Figure 2). Note that the convolution is easily evaluated as follows:

$$\nabla^2(Q_{k_c} * u_{k_c}) = -\sum_{|\mathbf{k}|_\infty \le k_c} \hat{\nu}_t(|\mathbf{k}|_\infty)|\mathbf{k}|^2 \hat{\mathbf{u}}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.5}$$

It is important to note at this point that the eddy-viscosity model is linear but for a weak nonlinear dependency of $\nu_t(k_c)$ on $\mathbf{u}_{k_c}$ through $E(k_c)$ (see (5.3)).

Many authors have elaborated on this theory, among which are Chollet and Lesieur [10], who proposed exponential forms of the cusp and power laws like

$$\hat{\nu}_t(|\mathbf{k}|_\infty) = \left[\phi_0 + \phi_1 |\mathbf{k}|_\infty^n k_c^{-n}\right], \tag{5.6}$$

where ϕ_0, ϕ_1 and n are *ad hoc* constants to be determined (see also Lesieur and Métais [58, 49]). The reader is also referred to [14, 65] for reviews of these models.

In order to cast a new light upon this type of models, we now recall the spectral viscosity technique of Tadmor [72] for approximating nonlinear conservation laws. We shall return to Kraichnan's eddy-viscosity model in Section 5.3..

5.2. Tadmor's spectral viscosity

The concept of spectral viscosity introduced by Tadmor [72] is a very powerful tool originally designed to prove convergence of spectral approximations of nonlinear conservation laws.

Let us consider a scalar conservation law in the d-dimensional torus $\Omega = (0, 2\pi)^d$ augmented with the entropy condition,

$$\begin{cases} u|_{t=0} = u_0 \in L^\infty(\Omega), \\ \partial_t u + \nabla\cdot \mathbf{f}(u) = 0, \quad \text{in the distributional sense}, \\ \partial_t U(u) + \nabla\cdot \mathbf{F}(u) \le 0, \quad \forall\ U \text{ convex}, \ \mathbf{F}(u) = \int_0^u U'(w)\mathbf{f}'(w)dw, \end{cases} \tag{5.7}$$

[1] Although in the 3D torus $L = 2\pi$, we keep the ratio $L/2\pi$ to keep track of dimensions.

where $\mathbf{f}$ is a flux a class $\mathcal{C}^s$, s being sufficiently large.

In order to build a spectral approximation of (5.7), we introduce $k_c > 1$ and we denote by P_{k_c} the L^2-projection onto k_c-trigonometric polynomials. The standard Galerkin technique consists in projecting (5.7) onto these k_c-trigonometric polynomials. The resulting scheme is spectrally accurate, but Gibbs oscillations triggered by shocks prevent the approximate solution to converge in $L^1(\Omega)$ to the entropy solution; that is, to the solution that satisfies the entropy conditions in (5.7). To suppress these oscillations while retaining spectral accuracy, Tadmor [72] proposed to augment the Fourier approximation by a spectral viscosity as follows. We look for $u_{k_c}(\mathbf{x},t) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, so that

$$\begin{cases} \partial_t u_{k_c} + \nabla\cdot\left(P_{k_c}\mathbf{f}(u_{k_c}(\mathbf{x},t)\right) = \varepsilon_{k_c}\nabla^2(Q_{k_c} * u_{k_c}(\mathbf{x},t)), \\ u_{k_c}|_{t=0} = P_{k_c}u_0. \end{cases} \tag{5.8}$$

where the vanishing viscosity ε_{k_c} and the viscosity kernel Q_{k_c} satisfies the following conditions:

T1. There is some $0 < \theta < 1$, such that the coefficient ε_{k_c} satisfies

$$\varepsilon_{k_c} \sim {k_c}^{-\theta}, \quad \text{and} \quad \varepsilon_{k_c}^s \|\partial_x^s P_{k_c} u_0\|_{L^2(\Omega)} \le \text{const}, \quad \forall\ s \ge 0. \tag{5.9}$$

T2. There exists a real number $k_i \gg 1$

$$k_i \sim \frac{{k_c}^{\frac{\theta}{2}}}{(\log k_c)^{\frac{d}{2}}}, \tag{5.10}$$

such that in the range $0 \le |\mathbf{k}|_\infty \le k_i$ there is no artificial viscosity.

T3. The viscosity kernel is given by its Fourier expansion $Q_{k_c}(\mathbf{x},t) = \sum_{|\mathbf{k}|_\infty = k_i}^{k_c} \hat{\mathbb{Q}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$, and the coefficients are spherically symmetric, that is, $\hat{\mathbb{Q}}_{\mathbf{k}} = \hat{Q}_p$ for all $|\mathbf{k}|_\infty = p$. Furthermore, the coefficients are monotonically increasing with respect to p and satisfy the estimate

$$|\hat{Q}_p - 1| \le c\frac{k_i^2}{p^2}, \qquad \forall\ p \ge k_i. \tag{5.11}$$

Note that viscosity coefficients like

$$\hat{Q}_{|\mathbf{k}|_\infty} = 1 - \frac{{k_i}^2}{\max(k_i, |\mathbf{k}|_\infty)^2}, \tag{5.12}$$

are acceptable, though it is probably better to have $\mathcal{C}^\infty$ smoothness with respect to k in applications (see [55, p. 336] and §5.3.). Note also that in applications one should use $\theta = 1^-$, for this yields best accuracy, though the limit case $\theta = 1$ is theoretically excluded. We also remark that the artificial viscosity term is easy to implement in the Fourier space since

$$\nabla^2(Q_{k_c} * u_{k_c}) = -\sum_{|\mathbf{k}|_\infty = k_i}^{k_c} \hat{Q}_{|\mathbf{k}|_\infty} |\mathbf{k}|^2 \hat{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.13}$$

and that this operator is linear.

The major result proved in [72] for the 1D case and in [8] for the d-dimensional case is as follows.

Theorem 5.1. *Under above assumptions, u_{k_c} converges to the unique entropy solution of* (5.7) *and the following error estimate holds:*

$$\|u - u_{k_c}\|_{L^\infty([0,T];L^1(\Omega))} \leq c\sqrt{\varepsilon_{k_c}}. \tag{5.14}$$

The fact that Tadmor's approach maintains spectral accuracy for nonlinear conservation laws and is able to select entropy solutions has prompted some authors to use the spectral viscosity operator to perform LES. For preliminary attempts in this direction we refer to Karamanos and Karniadakis [38], Pasquetti and Xu [60], and Adams and Stolz [1]. The goal of the next section is to compare Kraichnan's eddy-viscosity and the spectral viscosity regularization. We want to assess their respective accuracy and to determine whether these techniques regularize the Navier–Stokes equations.

5.3. Eddy-viscosity versus spectral viscosity

When comparing (5.2) and (5.8) we observe some similarity in the two approaches. In particular, the way the artificial viscosity operator is implemented in (5.5) and (5.13) is identical in both cases. The major difference lies in the scaling of the viscosities (5.3) and (5.9), and in the definition of the non-dimensional viscosity laws (5.6) and (5.12).

5.3.1. Graphic representation of $\hat{Q}_{|\mathbf{k}|_\infty}$ and $\hat{\nu}_t(|\mathbf{k}|_\infty)$

In order to get a better insight of the differences between the spectral viscosity and the eddy-viscosity, let us compare the graphs of two possible definitions of $\hat{Q}_{|\mathbf{k}|_\infty}$ and $\hat{\nu}_t(|\mathbf{k}|_\infty)$.

For the purpose of comparison we consider the following definition for Tadmor's spectral viscosity

$$\hat{Q}_{|\mathbf{k}|_\infty} = \tfrac{1}{2}(\tanh(\lambda x) + 1)\left[1 - \frac{1}{f(x)^2}\right] \tag{5.15}$$

where $x = \frac{|\mathbf{k}|_\infty - k_i}{k_i}$, $f(x) = \frac{1}{2}(\sqrt{\varepsilon + x^2} + x) + 1$, $\varepsilon = 0.01$ and $\lambda = \frac{1}{2}\log(\varepsilon/2\varepsilon_m)$ with ε_m denoting the machine accuracy. Note that this definition is slightly different from what is required in assumption T2 in the sense that if $|\mathbf{k}|_\infty < k_i$ then $\hat{Q}_{|\mathbf{k}|_\infty}$ is not zero, but this definition implies that $\hat{Q}_{|\mathbf{k}|_\infty}$ goes exponentially fast to zero when $|\mathbf{k}|_\infty$ is significantly smaller that k_i. This type of behavior is meant for $\hat{Q}_{|\mathbf{k}|_\infty}$ to be a smooth function of $|\mathbf{k}|_\infty$ on the entire range $0 \leq |\mathbf{k}|_\infty \leq k_c$ while being exponentially negligible for $|\mathbf{k}|_\infty < k_i$. We show in Figure 2 the graph of

$\hat{Q}_{|\mathbf{k}|_\infty}/\hat{Q}_{k_c}$ as a function of k/k_c for various cutoff numbers k_c using $\varepsilon_m = 10^{-16}$, $\theta = 1$, and $k_i = 5\sqrt{k_c}$.

For the eddy-viscosity $\hat{\nu}_t(|\mathbf{k}|_\infty)$, we choose one of the expressions proposed by Chollet and Lesieur [10] (see also [44, 49, 50]):

$$\hat{\nu}_t(|\mathbf{k}|_\infty) = (2.1)^{-3/2}\big[0.441 + 15.2\exp(-3.03\,k_c/|\mathbf{k}|_\infty)\big]. \tag{5.16}$$

The graphs of the normalized functions $\hat{Q}_{|\mathbf{k}|_\infty}/\hat{Q}_{k_c}$ and $\hat{\nu}_t(|\mathbf{k}|_\infty)/\hat{\nu}_t(k_c)$ are represented in Figure 2 as functions of k/k_c for $k_c = 64$ and $k_c = 128$.

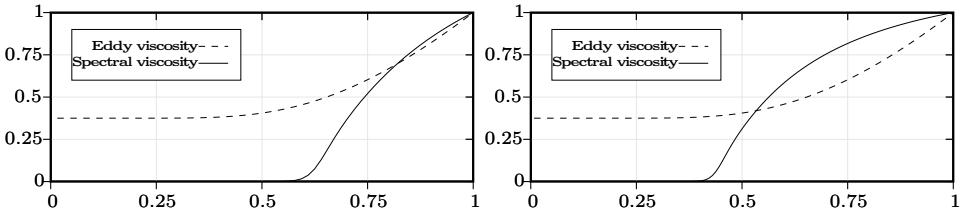

FIG. 2. Spectral viscosity versus Kraichnan–Chollet–Lesieur's model. Left: $k_c = 64$. Right: $k_c = 128$.

5.3.2. Spectral accuracy

The spectral viscosity $\hat{Q}_{|\mathbf{k}|_\infty}$ converges exponentially to zero for $|\mathbf{k}|_\infty \leq k_i$, and we recall that this feature is meant for the spectral viscosity method to achieve spectral accuracy [8, 55, 72].

Contrary to the spectral viscosity, $\hat{Q}_{|\mathbf{k}|_\infty}$, the graph of the eddy-viscosity, $\hat{\nu}_t(|\mathbf{k}|_\infty)$, does not vanish in the low mode region. This feature *de facto* destroys spectral accuracy of the numerical method. To better appreciate this argument, let us come back to definition (5.3). Assuming that the cut-off is realized in the inertial range of the Kolmogorov cascade, then $E(k_c) \sim k_c^{-5/3}$. If ϕ_0 in (5.6) is not zero, then (5.3) implies that $\nu_t(k_c)\hat{\nu}_t(\mathbf{k})$ can be bounded from below by $c\phi_0 k_c^{-4/3}$ and from above by $c(\phi_0 + \phi_1)k_c^{-4/3}$. Hence, except for the weakly nonlinear dependency of $\hat{\nu}_t(|\mathbf{k}|_\infty)$ on $\mathbf{u}_{k_c}$ through $E(k_c)$, we infer that, in first approximation, Kraichnan–Chollet–Lesieur's eddy-viscosity model is a standard (almost constant and almost linear) artificial viscosity scaling like $k_c^{-4/3}$. This result, which does not seem to be stressed in the literature, is odd. Formally, it should limit the convergence of the method to $\mathcal{O}(k_c^{-4/3})$, which is by far not spectral.

Let us now recall an interesting result from [8].

Lemma 5.1. *Under hypotheses* T1, T2, T3, *there exists a constant* c *independent of* k_c *s.t.*

$$\forall\, p \geq 1, \qquad \sup_{v_{k_c}\neq 0} \varepsilon_{k_c} \frac{\|\nabla^2(Q_{k_c} * v_{k_c}) - \nabla^2 v_{k_c}\|_{0,p}}{\|v_{k_c}\|_{0,p}} \leq c.$$

This lemma states that the spectral viscosity is an L^p-bounded perturbation of the standard first-order linear viscosity (the lemma remaining true for $\theta = 1$ as well). Since, as shown above, the eddy-viscosity is an almost constant first-order linear viscosity, this result means that the spectral viscosity acts in a similar manner on the high wavenumber modes as the eddy-viscosity, the only difference being that it does not spoil the consistency error by unnecessarily dampening the low wavenumber modes. In other words, this argument proves that, modulo the scaling factor discussed above, the spectral viscosity really works like the eddy-viscosity while retaining spectral accuracy.

In the literature, the fact that $\hat{\nu_t}(|\mathbf{k}|_\infty)$ is not zero when $\mathbf{k} \to 0$ is meant to account for long range interactions between very small scales and large scales (*i.e.* the so-called nonlocal triadic interactions). Although we do not feel qualified to assess the relevance of this argument, we observe that it seems to be definitely incompatible with spectral accuracy, thus questioning the use of spectral methods in this context.

5.3.3. The DNS point of view

It is reasonable to think that Direct Numerical Simulations may bring some clue on the validity of laws like (5.6) and (5.16).

Let us assume that DNS data are at hand. Let us denote by k_∞ the cutoff for this DNS. Upon denoting $\mathbf{u}_{k,k'} = \sum_{|\boldsymbol{\ell}|_\infty=k}^{k'} \hat{\mathbf{u}}(\boldsymbol{\ell},t)e^{i\boldsymbol{\ell}\cdot\mathbf{x}}$, the usual procedure to evaluate a, possibly time-dependent, eddy-viscosity $\nu_t(k,t)$ is as follows

$$\nu_t(k,t) = \frac{\int_\Omega (\mathbf{u}_{k_\infty}\cdot\nabla\mathbf{u}_{k_\infty} - \mathbf{u}_{k_c}\cdot\nabla\mathbf{u}_{k_c})\cdot\mathbf{u}_{k,k}\mathrm{d}\mathbf{x}}{2k^2E(k,t)}, \tag{5.17}$$

where $E(k,t)$ is the (possibly time-dependent) kinetic energy at the wavenumber k. This function is such that

$$\left[\partial_t + 2k^2(\chi(k)\nu + \nu_t(k,t))\right] E(k,t) = \int_\Omega (\mathbf{f} - \mathbf{u}_{k_c}\cdot\nabla\mathbf{u}_{k_c})\cdot\mathbf{u}_{k,k}\mathrm{d}\mathbf{x},$$

where $\chi(k) = \frac{1}{2}\sum_{|\boldsymbol{\ell}|_\infty=k} |\boldsymbol{\ell}|^2|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2/k^2E(k,t)$ is almost constant if the flow is isotropic.

Of course, assuming that $\nu_t(k,t)$ (or some time average of it) is the one that models reasonably the missing nonlinear terms in the momentum equation is an act of faith. Actually, as reported in [57], "In general, however, we can state quite categorically that, for the case of a sharp cutoff in wavenumber space, no eddy-viscosity model (subject to the reasonable constraint $\nu_t(k,t) \geq 0$) can perfectly reproduce the missing nonlinear terms."

Nevertheless, we show in Figure 3 the ratio $\langle\nu_t(k,t)\rangle/\nu$ for a stationary flow in the 3D-torus, where $\langle\cdot\rangle$ is the time average. The DNS simulations have been performed by McComb and Young and are reported in [57, Fig. 15]. The DNS uses 128^3 modes. In Figure 3, the authors have plotted the ratio $\nu_t(k)/\nu$ as a function of k/k_c for various cutoff wavenumbers $k_c \ll 128$.

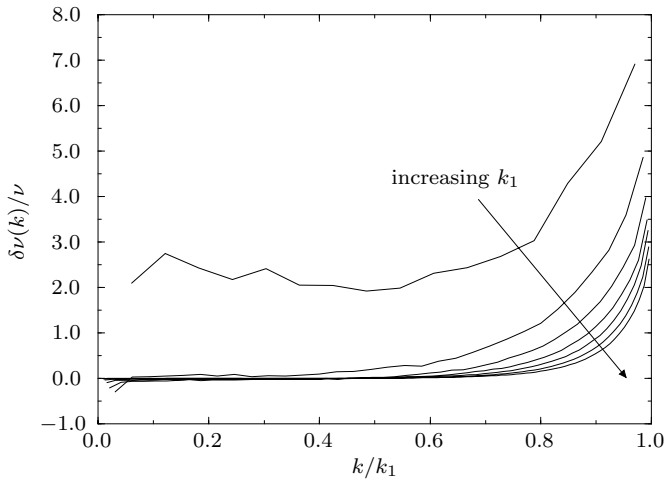

FIG. 3. Spectral viscosity from McComb and Young [57, Fig. 15], ν_t/ν vs. k/k_c for various cutoffs $k_c = 16$, 32, 48, 64, 80, 96, 112 (Courtesy of D. McComb).

These results seem both to confirm (see $k_c = 16$) and to contradict (see $32 \leq k_c \leq 112$) laws like (5.6) and (5.16). The case $k_c = 16$ is in perfect agreement with (5.6), but this cutoff is so low that it is hard to believe that any significant part of the spectrum is really resolved. This curve clearly shows that all the scales of the spectrum interact with each other. This is in perfect agreement with the fact that laws like (5.6) and (5.16) are meant to reproduce nonlocal energy transfers at all scales. We conclude from this observation that laws like (5.6) or (5.16) are well suited to situations where no significant part of the energy spectrum is resolved, *i.e* under-resolved LES. For the numerical point of view, under-resolved LES with an eddy-viscosity like (5.16) can be done with any second-order numerical technique, since upon setting $k_c = 1/h$ and assuming $E(k_c) \sim k_c^{-5/3}$, one obtains an eddy-viscosity of order $h^{4/3}$ which is clearly significantly larger than the h^2 consistency error of the scheme. In conclusion, it seems to us that the use of spectral methods to perform under-resolved LES may be unnecessary since spectral accuracy cannot be achieved.

If one really wants to use spectral methods to do LES and if one really expects to achieve spectral accuracy, then the cutoff wavenumber should be chosen large enough for the large scales to be resolved. Whether such an expectation is physically realistic we shall let the turbulence specialists decide. Nevertheless, returning to Figure 3, it seems reasonable to expect that for $k_c \geq 32$ the lower part of the spectrum of the velocity field is somewhat resolved. The figure suggest that in the low mode region the computed eddy-viscosity goes to zero extremely fast as k_c grows. The convergence rate to zero seems to be much faster than $k_c^{-4/3}$ which is the rate produced by the law (5.6). Hence, eddy-viscosity models like (5.6) do

not seem to be compatible with LES for which the lower part of the spectrum is resolved. Note also that Figure 3 corroborates the spectral viscosity postulate that the artificial viscosity should be zero in the low mode.

Finally concerning the cusp form of the curves in Figure 3, some agreement among turbulence specialists seems to emerge in the literature to state that it may be an artifact that comes from the particular definition (5.17) that is assumed for $\nu_t(k)$. We refer to Quarini and Leslie [51] for a thorough discussion of this phenomenon. To conclude on this matter, we repeat that the definition (5.17) is an act of faith, since it cannot be proved that eddy-viscosity models can perfectly represent the contribution of the nonlinear terms that are not accounted for in numerical simulations.

5.3.4. The well-posedness issue

One may finally wonder whether a spectral viscosity perturbation of the Navier–Stokes equation could solve the uniqueness issue in the same way as the Leray regularization and Ladyženskaja's nonlinear viscosity did. In other words, does a spectral viscosity perturbation regularize the Navier–Stokes equations?

To answer this question, let us extend the operator Q_{k_c} to $L^1(\Omega)$ by defining $\tilde{Q}_{k_c} = Q_{k_c} + \sum_{|\mathbf{k}|_\infty > k_c} e^{i\mathbf{k}\cdot\mathbf{x}}$. Formally, $\tilde{Q}_{k_c} = \delta + \sum_{|\mathbf{k}|_\infty \le k_c} (\hat{\nu}_t(|\mathbf{k}|_\infty) - 1) e^{i\mathbf{k}\cdot\mathbf{x}}$, where δ is the Dirac measure. Let $\mathbf{P}_\infty$ be the $\mathbf{L}^2$-projection onto $\mathbf{H}$, and let us modify the Navier–Stokes equations as follows

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon - \mathbf{P}_\infty \nabla^2 \big(\nu \mathbf{u}_\varepsilon + \varepsilon \tilde{Q}_{k_c} * \mathbf{u}_\varepsilon\big) + \mathbf{P}_\infty (\mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon) = \mathbf{P}_\infty \mathbf{f}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0, \end{cases} \tag{5.18}$$

where ε is some vanishing viscosity. Then, it is clear that

$$\forall\ \mathbf{v} \in (1 - \mathbf{P}_{k_c}) \mathbf{H}_0^1(\Omega), \qquad \nabla^2 \left[\nu \mathbf{v} + \varepsilon \tilde{Q}_{k_c} * \mathbf{v}\right] = (\nu + \varepsilon) \nabla^2 \mathbf{v}.$$

That is, except for the finite-dimensional vector space of the k_c-trigonometric solenoidal polynomials, adding a spectral viscosity just amounts to replacing the viscosity ν by $\nu + \varepsilon$, which is clearly not enough for solving the uniqueness question (at the present time). Hence, modifying the Navier–Stokes equations by adding a linear spectral viscosity does not resolve the uniqueness problem.

Note that we cannot blindly extend directly to the eddy-viscosity the negative conclusion we have drawn for the linear spectral viscosity, for eddy-viscosity models are slightly nonlinear through their dependency on $E(k_c)$. However, owing to the evident uniform boundedness of $\|\mathbf{u}(t)\|_0^2$, it is clear that $E(k_c)$ is bounded uniformly in time, which implies that $\nu_t(k_c)$ is also uniformly bounded in time. Now let us consider the Navier–Stokes equations modified by adding a spectral eddy-viscosity that has been extended beyond k_c to all the Fourier modes in $\mathbb{Z}^3$ as

above. In view of what was proved earlier, adding a spectral eddy-viscosity to the Navier–Stokes equation is equivalent to adding a time-dependent constant to the Reynolds number for all the modes beyond k_c. Even though the variations in the Reynolds number may be time-dependent, their amplitude is bounded uniformly in time. Hence, from what we know from the Navier–Stokes equation, it is doubtful that eddy-viscosity model can solve the uniqueness question. It is sometimes claimed in literature that Kraichnan–Chollet–Lesieur's viscosity is "equal to" the Smagorinsky model (see *e.g.* Lesieur [48, p. 237]). In view of the radically different behavior of both techniques concerning the uniqueness issue, we found this claim dubious.

At this point we realize that one must temper the hope that one might have to give some mathematical justification to eddy-viscosity methods by invoking the fact that spectral viscosities select entropy solutions of nonlinear scalar conservation laws. Although the Navier–Stokes equations and nonlinear scalar conservation laws share some important features, these two sets of PDE's are quite different. The difference is rooted in the fact that adding a linear first-order viscosity to nonlinear scalar conservation laws is just what it takes to select the entropy solution, whereas decreasing the Reynolds number in the Navier–Stokes equations does not resolve the well-posedness problem.

5.3.5. Proposition for a nonlinear spectral viscosity

From the previous section it is clear that eddy-viscosity and spectral viscosity techniques do not regularize the Navier–Stokes equations. The reason is that these operators are linear or quasi-linear and of second-order only. We now propose a spectral hyper-viscosity that should resolve this problem.

Let us set $k_i = k_c^\theta$ and let us define $\hat{Q}_k$ as in (5.15), (5.12), or (5.13). Now, having in mind the iterated Laplacian operator, we define

$$\hat{\nu}_t(\mathbf{k}, \mathbf{u}) = \hat{Q}_{|k|_\infty} |\mathbf{k}|^\alpha, \quad \frac{5}{4} < \alpha, \tag{5.19}$$

and

$$\nu_t(k_c) = k_c^{-2\theta\alpha}, \quad 0 < \theta < 1 - \frac{5}{4\alpha}. \tag{5.20}$$

Then the nonlinear viscosity kernel assumes the following form

$$Q_{k_c}(\mathbf{x}, \mathbf{u}) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{\nu}_t(\mathbf{k}, \mathbf{u}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.21}$$

Although a thorough mathematical analysis of this model is out of question here, the reader may convince himself that this kernel yields spectral accuracy, and, since it yields an *a priori* estimate in $L^2(0, T; \mathbf{H}^\alpha(\Omega))$, that it actually provides for a regularization of the Navier–Stokes equations.

6. Two-scale methods

Hierarchical multilevel settings are very often credited for providing suitable frameworks for LES. We review this point of view in the present section, which is organized as follows: first, we describe popular two-scale subgrid viscosity models; then, we review scale similarity models; finally, we show that numerical implementations of some two-scale models are very similar to a subgrid stabilization technique that has been introduced in the literature to solve non-coercive PDE's, [25, 27, 26].

This investigation led us to draw the following conclusions:

(i) Standard scale-similarity models do not yield coherent energy estimates.

(ii) A new scale-similarity model introduced in [45] yields coherent energy estimates, but a discrete version of the model cannot realistically satisfy all assumptions, namely the discrete incompressibility, the commutation of the filter with differential operators, and the self-adjointness of the filter.

(iii) A new scale-similarity model that solves the problems raised in (ii) can be derived.

(iv) There seem to be no mathematical results, to our best knowledge, that would show that two-scale models are regularization techniques.

(v) The only mathematical result in support of this class of methods that we have been able to identify so far is that, when approximating linear noncoercive PDE's, the addition of a two-scale subgrid viscosity guarantees optimal convergence in the graph norm of the approximate solution.

Throughout this section, we assume that we have at hand two finite dimensional spaces, $\mathbf{X}_h$ and M_h, for approximating the velocity and the pressure, respectively. To avoid irrelevant stability issues, we assume that $\mathbf{X}_h$ and M_h satisfy the LBB condition. We assume also that we are given a linear operator $\mathbf{P}_H : \mathbf{X}_h \longrightarrow \mathbf{X}_h$ that we shall refer to as a filter. We call $\mathbf{X}_h$ the fine scale space and $\mathbf{X}_H = \mathbf{P}_H(\mathbf{X}_h)$ the resolved scale space. Here H and h refer to the characteristic meshsizes of $\mathbf{X}_H$ and $\mathbf{X}_h$, respectively. It is common in the literature to take $H \approx 2h$. We finally suppose that $\mathbf{X}_H$ satisfies interpolation properties of $\mathcal{O}(H^{k+1})$ in $\mathbf{L}^2(\Omega)$ and of $\mathcal{O}(H^k)$ in $\mathbf{H}^1(\Omega)$.

6.1. Two-scale subgrid viscosity methods

The robustness of artificial viscosity techniques has led authors to adapt this class of methods to the two-scale approximation framework. For two-scale subgrid viscosity methods, the main motivation is to construct a semilinear form $a_{\text{sgs}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h)$ accounting for the modeling of a subgrid tensor and to solve the

following approximate Navier–Stokes equations:

$$\left\{\begin{array}{l} \text{Find } \mathbf{u}_h \in \mathcal{C}^1(0,T;\mathbf{X}_h) \text{ and } p_h \in \mathcal{C}^0(0,T;M_h) \text{ such that} \\ \quad (d_t\mathbf{u}_h,\mathbf{v}_h)+(\mathbf{u}_h\cdot\nabla\mathbf{u}_h,\mathbf{v}_h)-(p_h,\nabla\cdot\mathbf{v}_h)+\nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) \\ \qquad\qquad +a_{\rm sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h), \qquad \forall\mathbf{v}_h\in\mathbf{X}_h \\ \quad (q_h,\nabla\cdot\mathbf{u}_h)=0, \quad \forall q_h\in M_h \\ \quad \mathbf{u}_h|_{t=0}=\mathcal{I}_h\mathbf{u}_0, \end{array}\right. \tag{6.1}$$

where $\mathcal{I}_h$ is an $\mathbf{L}^2$-stable interpolation operator in $\mathbf{X}_h$.

6.1.1. The subgrid viscosity method

The strategy for these techniques consists in modeling the subgrid tensor in the form of a dissipative operator like $\nabla\cdot\mathbb{T}=\nabla\cdot(-\nu_t\nabla\mathbf{u})$ where the so-called turbulent viscosity, ν_t, is assumed to depend only on the fluctuating component of the velocity.

Adopting, formally for the time being, a decomposition of the velocity field in the form $\mathbf{u}=\overline{\mathbf{u}}+\mathbf{u}'$, one possibility for defining ν_t consists in assuming that the turbulent viscosity depends only on the turbulent kinetic energy $e'=\frac{1}{2}\mathbf{u}'^2$ and a turbulence mixing length scale ε s.t. $\nu_t\sim\varepsilon e'^{1/2}$. Numerically, this idea can be implemented using two-scale approximation techniques by identifying $\mathbf{u}$ with $\mathbf{u}_h$ and $\overline{\mathbf{u}}$ with $\mathbf{P}_H(\mathbf{u}_h)$; then, the discrete turbulent kinetic energy is $\frac{1}{2}(\mathbf{u}_h-\mathbf{P}_H(\mathbf{u}_h))^2$ and the turbulence length scale at hand is the coarse meshsize H. Hence, the turbulent kinetic energy model, usually referred to as the TKE model (see *e.g.* [74]), consists in setting

$$\nu_t\approx cH|\mathbf{u}_h-\mathbf{P}_H(\mathbf{u}_h)|. \tag{6.2}$$

Another possibility for defining ν_t consists in using a mixed approach combining the Smagorinsky model and the TKE eddy-viscosity introduced above as in Sagaut *et al.* [66]:

$$\nu_t(\mathbf{u}_h)=cH^{1+\alpha}|\mathbf{P}_H(\mathbf{u}_h)-\mathbf{u}_h|^{1-\alpha}\left\{\begin{array}{c}|\mathbf{D}(\mathbf{u}_h)|^\alpha \\ \text{or} \\ |\nabla\times\mathbf{u}_h|^\alpha\end{array}\right. \qquad 0\le\alpha\le 1. \tag{6.3}$$

We refer to Sagaut [65] for a thorough review on this class of models. One major pragmatic interest in this kind of formula is that it combines the stabilizing effects of the Smagorinsky model (for $\alpha>0$) while allowing for high order consistency. For instance, if the filter is such that for smooth solutions $\|\mathbf{P}_H(\mathbf{u}_h)-\mathbf{u}_h\|=\mathcal{O}(H^{k+1})$, then the subgrid term is $\mathcal{O}(H^{k(1-\alpha)+2})$, which is at most $\mathcal{O}(H^{k+1})$ if $\alpha\le k^{-1}$. In particular, in viscous layers, where the solution is smooth, the subgrid term is of the same order as the consistency error of the underlying numerical scheme.

In weak form, the subgrid semilinear form $a_{\rm sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)$ can be written:

$$a_{\rm sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)=(\nu_t(\mathbf{u}_h)\nabla\mathbf{u}_h,\nabla\mathbf{v}_h), \qquad \mathbf{u}_h,\ \mathbf{v}_h\in\mathbf{X}_h \tag{6.4}$$

where $\nu_t(\mathbf{u}_h)$ is given by either one of the two subgrid viscosity techniques described above.

6.1.2. The variational multiscale method

An alternative strategy proposed by Hughes *et al.* [32, 31, 33], called the LES "variational multiscale method," is based on the idea that the additional dissipation term should act only on the turbulent part of the velocity field and should leave the resolved part unchanged. This idea amounts to writing the subgrid tensor in the form $\nabla\cdot\mathbb{T} = (\nabla\cdot(-\nu_t\nabla\mathbf{u}'))'$. Within a two-scale approximation framework, if we assume that the fluctuation operator $(\cdot)'$ is self-adjoint and if we identify it to $(\mathbf{I}-\mathbf{P}_H)$, then the corresponding semilinear form that is added to the momentum equation is as follows:

$$a_{\text{sgs}}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) = \big(\nu_t(\mathbf{u}_h)\nabla(\mathbf{u}_h-\mathbf{P}_H(\mathbf{u}_h)),\nabla(\mathbf{v}_h-\mathbf{P}_H(\mathbf{v}_h))\big), \tag{6.5}$$

where the turbulent viscosity may take various forms. In [32], the authors propose the following two possible choices:

$$\nu_t(\mathbf{u}_h) = cH^2\begin{cases}|\mathbf{D}(\mathbf{u}_h)|\\ \quad\text{or}\\ |\mathbf{D}(\mathbf{u}_h-\mathbf{P}_H(\mathbf{u}_h))|\end{cases} \tag{6.6}$$

From numerical tests reported in [31, 33], it is unclear which form of ν_t performs the best, but from the approximation point of view, we note that in the regions where the solution is smooth, the first form of the subgrid term is $\mathcal{O}(H^{k+1})$, which is exactly the order of the consistency error of the approximation method, whereas the other form is unnecessarily much smaller, *i.e.* $\mathcal{O}(H^{2k+1})$.

6.2. Scale similarity models

Besides approaches that consist in approximating the subgrid tensor as a dissipative operator, another class of techniques introduced by Bardina *et al.* [2] aims at modeling the subgrid tensor as a dispersive operator by assuming scale-similarity.

6.2.1. The standard scale-similarity model

Let $\mathcal{F} = \widetilde{(\cdot)}$ and $\mathcal{G} = \overline{(\cdot)}$ denote two filtering operators, possibly different from each other. The scale-similarity hypothesis leads to the model

$$\mathbb{T} = \widetilde{\mathbf{u}\otimes\mathbf{u}} - \widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}} \;\sim\; \overline{\widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}}} - \overline{\widetilde{\mathbf{u}}}\otimes\overline{\widetilde{\mathbf{u}}} \tag{6.7}$$

where the similarity holds up to a multiplicative constant. Denoting by $\varepsilon_{\mathcal{F}}$ and $\varepsilon_{\mathcal{G}}$ the length scales associated with the two filters, the hypothesis can be written [45] as follows:

> The eddies smaller than $\mathcal{O}(\varepsilon_{\mathcal{F}})$ interact with those $\mathcal{O}(\varepsilon_{\mathcal{F}})$ in the mean in the same way those $\mathcal{O}(\varepsilon_{\mathcal{F}})$ interact with those $\mathcal{O}(\varepsilon_{\mathcal{G}})$ in size.

In a discrete context involving a two-scale approximation strategy, one thinks of $\mathcal{F}$ as an interpolation operator on the grid of size h and of $\mathcal{G}$ as P_H. In this framework, assuming the similarity constant to be close to unity, the approximate subgrid tensor takes the following form:

$$\mathbb{T}_h(\mathbf{u}_h) = \mathbf{P}_H(\mathbf{u}_h \otimes \mathbf{u}_h) - \mathbf{P}_H(\mathbf{u}_h) \otimes \mathbf{P}_H(\mathbf{u}_h),$$

and the corresponding weak form is

$$a_{\text{sgs}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = (\nabla \cdot \mathbb{T}_h(\mathbf{u}_h), \mathbf{v}_h), \qquad \mathbf{u}_h \in \mathbf{X}_h,\ \mathbf{v}_h \in \mathbf{X}_h. \tag{6.8}$$

When compared to DNS, this model shows very high correlations with the actual subgrid tensor [2, 30]. However, in practice, according to Ferziger [14], "it is found that this model hardly dissipates any energy and cannot serve as a stand alone SGS model". In other words, when used alone, this model yields unsustainable numerical instabilities, the reason being that the semilinear form a_{sgs} is not positive. Layton [45] has analyzed this problem in more details and has proposed minor modifications to the model in order to obtain coherent energy estimates.

6.2.2. A modified scale-similarity model

Noting that the scale-similarity hypothesis can be restated as follows:

$$\widetilde{\mathbf{u} \otimes \mathbf{u}} \sim \widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}} + \overline{\widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}}} - \overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}}, \tag{6.9}$$

and using the notation $\widetilde{\mathbf{u}}' = \widetilde{\mathbf{u}} - \overline{\widetilde{\mathbf{u}}}$, so that $\widetilde{\mathbf{u}} = \overline{\widetilde{\mathbf{u}}} + \widetilde{\mathbf{u}}'$, Layton in [45] proposes to further develop the right-hand side of (6.9) such that

$$\widetilde{\mathbf{u} \otimes \mathbf{u}} \sim \overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}} + \overline{\overline{\widetilde{\mathbf{u}}} \otimes \widetilde{\mathbf{u}}' + \widetilde{\mathbf{u}}' \otimes \overline{\widetilde{\mathbf{u}}}} + \mathbf{R}(\widetilde{\mathbf{u}}),$$

where the residual $\mathbf{R}(\widetilde{\mathbf{u}})$ reads

$$\mathbf{R}(\widetilde{\mathbf{u}}) = \widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}} - \overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}} + \overline{\overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}}} - \overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}} + \overline{\widetilde{\mathbf{u}}' \otimes \widetilde{\mathbf{u}}'}.$$

The tensor $\mathbf{R}$ is then modeled using one of the dissipative subgrid viscosity models presented in the previous section 6.1.1., and we shall henceforth denote by $a_{\text{sgs}}(\mathbf{u}; \mathbf{u}, \mathbf{v})$ the corresponding semilinear weak form.

Now let us turn our attention to the remaining tensor. The original result pointed out in [45] is that this tensor is dispersive; *i.e.*, its contribution to the global kinetic energy balance is zero. More precisely, let us define the following semilinear form:

$$b(\widetilde{\mathbf{u}}; \widetilde{\mathbf{v}}) = \int_\Omega \nabla \cdot \left(\overline{\widetilde{\mathbf{u}}} \otimes \overline{\widetilde{\mathbf{u}}} + \overline{\overline{\widetilde{\mathbf{u}}} \otimes \widetilde{\mathbf{u}}' + \widetilde{\mathbf{u}}' \otimes \overline{\widetilde{\mathbf{u}}}} \right) \cdot \widetilde{\mathbf{v}} \, \mathrm{d}x.$$

Proposition 6.1. *Provided the linear filter* $\overline{[\cdot]}$ *is self-adjoint and commutes with differential operators, then for all smooth solenoidal vector field* $\widetilde{\mathbf{u}}$ *we have* $b(\widetilde{\mathbf{u}}; \widetilde{\mathbf{u}}) = 0$.

This property is essential to prove existence of solutions to the modified Navier–Stokes equations. Unfortunately, when it comes to discretize the above semilinear form one cannot expect the incompressibility constraint, the commutation, and the self-adjointness hypotheses to be satisfied exactly. We now suggest a slightly changed formulation in order to remedy with these difficulties.

6.2.3. Proposition for a new scale-similarity model

Our approach is to further modify the formulation by rewriting the transport term in rotational form using the identities (3.11). Invoking the fact that the velocity fields are solenoidal, we have

$$\nabla\cdot\left(\overline{\widetilde{\mathbf{u}}}\otimes\overline{\widetilde{\mathbf{u}}}\right)=\overline{\widetilde{\mathbf{u}}}\cdot\nabla\overline{\widetilde{\mathbf{u}}}=(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\overline{\widetilde{\mathbf{u}}}+\frac{1}{2}\nabla\overline{\widetilde{\mathbf{u}}}^2$$

and

$$\begin{aligned}\nabla\cdot\left(\overline{\widetilde{\mathbf{u}}}\otimes\widetilde{\mathbf{u}}'+\widetilde{\mathbf{u}}'\otimes\overline{\widetilde{\mathbf{u}}}\right)&=\widetilde{\mathbf{u}}'\cdot\nabla\overline{\widetilde{\mathbf{u}}}+\overline{\widetilde{\mathbf{u}}}\cdot\nabla\widetilde{\mathbf{u}}'\\&=(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\widetilde{\mathbf{u}}'+(\nabla\times\widetilde{\mathbf{u}}')\times\overline{\widetilde{\mathbf{u}}}+\nabla\left(\widetilde{\mathbf{u}}'\cdot\overline{\widetilde{\mathbf{u}}}\right).\end{aligned}$$

Then, the semilinear form $b(\cdot;\cdot)$ introduced above can be rewritten as follows:

$$\begin{aligned}b(\widetilde{\mathbf{u}};\widetilde{\mathbf{v}})=\int_\Omega\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\overline{\widetilde{\mathbf{u}}}\right]\cdot\widetilde{\mathbf{v}}+\overline{\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\widetilde{\mathbf{u}}'+(\nabla\times\widetilde{\mathbf{u}}')\times\overline{\widetilde{\mathbf{u}}}\right]}\cdot\widetilde{\mathbf{v}}\,\mathrm{d}x\\+\int_\Omega\nabla\big(\frac{1}{2}\overline{\widetilde{\mathbf{u}}}^2+\widetilde{\mathbf{u}}'\cdot\overline{\widetilde{\mathbf{u}}}\big)\cdot\widetilde{\mathbf{v}}\,\mathrm{d}x,\end{aligned}$$

Now, we observe that the filter involved in the second term on the right-hand side of above inequality, is not exactly $\mathcal{G}(\cdot)=\overline{\left[\,\cdot\,\right]}$ but rather its adjoint, so that we write

$$\begin{aligned}b(\widetilde{\mathbf{u}};\widetilde{\mathbf{v}})=\int_\Omega\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\overline{\widetilde{\mathbf{u}}}\right]\cdot\widetilde{\mathbf{v}}+\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\widetilde{\mathbf{u}}'+(\nabla\times\widetilde{\mathbf{u}}')\times\overline{\widetilde{\mathbf{u}}}\right]\cdot\overline{\widetilde{\mathbf{v}}}\,\mathrm{d}x\\+\int_\Omega\nabla\big(\frac{1}{2}\overline{\widetilde{\mathbf{u}}}^2+\widetilde{\mathbf{u}}'\cdot\overline{\widetilde{\mathbf{u}}}\big)\cdot\widetilde{\mathbf{v}}\,\mathrm{d}x.\end{aligned}$$

It is clear that the third term can be added to the pressure by redefining the total pressure as $\pi=p+\frac{1}{2}\overline{\widetilde{\mathbf{u}}}^2+\widetilde{\mathbf{u}}'\cdot\overline{\widetilde{\mathbf{u}}}$. Then, upon defining the semilinear form

$$c(\widetilde{\mathbf{u}};\widetilde{\mathbf{v}})=\int_\Omega\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\overline{\widetilde{\mathbf{u}}}\right]\cdot\widetilde{\mathbf{v}}+\left[(\nabla\times\overline{\widetilde{\mathbf{u}}})\times\widetilde{\mathbf{u}}'+(\nabla\times\widetilde{\mathbf{u}}')\times\overline{\widetilde{\mathbf{u}}}\right]\cdot\overline{\widetilde{\mathbf{v}}}\,\mathrm{d}x$$

we have the following property.

Proposition 6.2. *For all smooth vector field* $\widetilde{\mathbf{u}}$ *we have* $c(\widetilde{\mathbf{u}};\widetilde{\mathbf{u}})=0$.

In other words, the contribution to the kinetic energy balance of the sum of the transport term and the subgrid term is zero pointwise. Note that this property

does not depend on the value of the divergence of the vector field nor on any commutation property between the filters and the differential operators. This new modeling shortcuts ongoing debates on the commutation problem.

At the discrete level we replace $\widetilde{\mathbf{u}}$ by $\mathbf{u}_h$ and $\overline{\widetilde{\mathbf{u}}}$ by $\mathbf{u}_H = P_H(\mathbf{u}_h)$. By denoting the subgrid component of $\mathbf{u}_h$ by $\mathbf{u}_h^H = \mathbf{u}_h - \mathbf{u}_H$, we introduce

$$d_{\text{sgs}}(\mathbf{u}_h; \mathbf{v}_h) = \big((\nabla \times \mathbf{u}_H) \times \mathbf{u}_h^H + (\nabla \times \mathbf{u}_h^H) \times \mathbf{u}_H, \mathbf{P}_H(\mathbf{v}_h)\big).$$

Then, the discrete two-scale Navier–Stokes problem can be recast in its final form as follows:

$$\begin{cases} \text{Find } \mathbf{u}_h \in \mathcal{C}^1(0,T;\mathbf{X}_h) \text{ and } p_h \in \mathcal{C}^0(0,T;M_h) \text{ such that,} \\ (d_t\mathbf{u}_h, \mathbf{v}_h) + \big((\nabla \times \mathbf{u}_H) \times \mathbf{u}_H, \mathbf{v}_h\big) - (\pi_h, \nabla \cdot \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) \\ \qquad\qquad + d_{\text{sgs}}(\mathbf{u}_h; \mathbf{v}_h) + a_{\text{sgs}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{X}_h \\ (q_h, \nabla \cdot \mathbf{u}_h) = 0, \quad \forall q_h \in M_h \\ \mathbf{u}_h|_{t=0} = \mathcal{I}_h \mathbf{u}_0, \end{cases} \tag{6.10}$$

where π_h is an approximation of the total pressure.

6.3. Subgrid viscosity stabilization

The goal of this section is to give a partial mathematical justification for the two-scale LES discrete models introduced above. We show that these models are closely related to a class of stabilizing techniques that solves non-coercive PDE's. More precisely, we show that adding a two-scale subgrid viscosity to the Galerkin formulation yields a method that is optimal for approximating linear contraction semi-groups of class $\mathcal{C}^0$. Throughout this section we shall use the following definition:

Definition 6.1 (Graph of a function and graph norm)**.** Let E and F two normed spaces and let $f : E \longrightarrow F$. The graph of f is defined as the subset of $E \times F$ such as

$$\text{graph } f = \{(x, f(x));\ x \in E\}.$$

The graph norm on E, associated with the function f, is defined as:

$$\|x\|_f = \|x\|_E + \|f(x)\|_F$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ are norms defined on E and F respectively.

6.3.1. The continuous setting

To illustrate our point of view, we consider hereafter the following linear problem. For $f \in \mathcal{C}^1([0,+\infty[;L)$ and $u_0 \in D(A)$,

$$\begin{cases} \text{find } u \in \mathcal{C}^1([0,+\infty[;L) \cap \mathcal{C}^0([0,+\infty[;D(A)) \text{ such that,} \\ u_{|t=0} = u_0, \\ d_t u + Au = f, \end{cases} \tag{6.11}$$

where L is a separable Hilbert space and $A : D(A) \subset L \longrightarrow L$ is a linear operator. We assume that A is monotone:

$$\forall v \in D(A), \qquad (Av, v)_L \geq 0, \tag{6.12}$$

and A is maximal

$$\forall f \in L,\ \exists v \in D(A), \qquad v + Av = f. \tag{6.13}$$

The reader may take $L = L^2(\mathbb{R})$ and of $A = \partial_x$ for simplicity. Now let us set $V = D(A)$ and let us equip V with the graph norm: $\|v\|_V = \|v\|_L + \|Av\|_L$. It can be shown that, the graph of A being closed, V is a Hilbert space when equipped with the scalar product $(u,v)_L + (Au, Av)_L$. For the sake of clarity in the presentation, we assume hereafter that V is a space of vector-valued functions on Ω in $\mathbb{R}^m$, $m \geq 1$.

Owing to the Hille–Yosida theorem (see *e.g.* Brezis [4, p. 110] or Yosida [75, p. 248]), problem (6.11) is well-posed and admits the following stability properties

$$\begin{cases} \|u\|_{\mathcal{C}^0([0,T];L)} \leq c\,(\|u_0\|_L + T\|f\|_{\mathcal{C}^0([0,T];L)}), \\ \|u\|_{\mathcal{C}^1([0,T];L)} + \|u\|_{\mathcal{C}^0([0,T];V)} \leq c\,(\|u_0\|_V + T\|f\|_{\mathcal{C}^1([0,T];L)}). \end{cases} \tag{6.14}$$

The fact that A is maximal is a key ingredient for proving the well-posedness of (6.11). This property can be better understood in the light of the following proposition:

Proposition 6.3. *Let $E \subset F$ be two Hilbert spaces with dense and continuous embedding, and let $A \in \mathcal{L}(E;F)$ be a monotone operator. The following two properties are equivalent.*

(i) *A is maximal.*

(ii) *There exists two constants $c_1 > 0$, $c_2 \geq 0$ such that*

$$\forall u \in E, \qquad \sup_{v \in F} \frac{(Au, v)_F}{\|v\|_F} \geq c_1 \|u\|_E - c_2 \|u\|_F. \tag{6.15}$$

The key to the theory developed herein is to build a discrete framework for which a discrete counterpart to (6.15) holds.

When it comes to approximating the solution to (6.11), it is known that the Galerkin technique is not appropriate if A is not coercive. In general, it is not possible to guarantee optimal convergence in the graph norm, since the discrete

counterpart of (6.15) is usually not satisfied uniformly with respect to the mesh-size. As a consequence, when approximating this type of equation supplemented with non-smooth data, the approximate solution exhibits spurious node-to-node oscillations. The two-level subgrid viscosity technique developed in Guermond [25, 27, 26] is one possible cure to this problem.

6.3.2. The discrete setting

Let us introduce three finite dimensional spaces X_h, X_H, X_h^H such that

$$V \supset X_h = X_H \oplus X_h^H. \tag{6.16}$$

We assume that X_h, X_H have suitable interpolation properties; that is, there is a dense subspace $W \subset V$ together with a linear interpolation operator $I_H \in \mathcal{L}(W; X_H)$ and two constants $k > 0$, $c > 0$ such that

$$\forall H,\ \forall v \in W, \quad \|v - I_H v\|_L + H\|v - I_H v\|_V \leq cH^{k+1}\|v\|_W. \tag{6.17}$$

One may view X_h as a fine scale space, X_H a coarse scale space and X_h^H a subgrid scale space where basis functions are highly fluctuating.

Denoting by h and H the meshsizes on which X_h and X_H are built respectively, we assume that H and h are of the same order; *i.e.* $c_1 h \leq H \leq c_2 h$. In practice we shall always use $H = 2h$. As a result, X_h being finite dimensional, we assume that there exists $c_i > 0$, independent of h and H, such that

$$\forall v_h \in X_h, \qquad \|v_h\|_V \leq c_i H^{-1}\|v_h\|_L. \tag{6.18}$$

Note that this hypothesis indirectly implies that A is a first order differential operator and justifies the assumption $c_1 h \leq H \leq c_2 h$.

We define $P_H : X_h \longrightarrow X_H$ as being the projection of X_h onto X_H that is parallel to X_h^H. We assume that

$$P_H \text{ is stable in the norm of } L \text{ uniformly w.r.t } H \text{ and } h. \tag{6.19}$$

For all v_h in X_h we denote

$$v_H = P_H v_h \qquad \text{and} \qquad v_h^H = (1 - P_H)v_h.$$

We introduce $b_h \in \mathcal{L}(X_h^H, X_h^H)$ s.t. for all $(v_h^H, w_h^H) \in X_h^H \times X_h^H$,

$$b_h(v_h^H, w_h^H) = H \int_\Omega \nabla v_h^H \cdot \nabla w_h^H \, \mathrm{d}x. \tag{6.20}$$

Note that b_h is a dissipative bilinear form. It is a subgrid viscosity that acts only on the subgrid scales. This property has to be put in parallel with the subgrid model (6.6) advocated in Hughes *et al.* [32] for which the theory discussed herein may be regarded as a partial justification.

Now we introduce the main hypothesis of this section, *i.e.* we assume that a discrete version of (6.15) is satisfied. More precisely, we assume that there are $c_1 > 0$ and $c_2 \geq 0$, independent of (H, h), such that

$$\forall v_h \in X_h, \quad \sup_{\phi_h \in X_h} \frac{(Av_H, \phi_h)_L}{\|\phi_h\|_L} \geq c_1\|v_H\|_V - c_2\|v_h\|_L. \tag{6.21}$$

Let us assume that $u_0 \in W$ so that u_0 can be approximated by $I_H u_0$. Then, the discrete problem we consider is

$$\begin{cases} \text{Find } u_h \in \mathcal{C}^1([0,+\infty[;X_h) \text{ s.t.} \\ (d_t u_h, v_h)_L + (Au_h, v_h)_L + b_h(u_h^H, v_h^H) = (f, v_h)_L, \qquad \forall v_h \in X_h, \\ u_{h|t=0} = I_H u_0. \end{cases} \tag{6.22}$$

This problem has a unique solution, for it is a system of linear ordinary differential equations. The major convergence result of this section is the following.

Theorem 6.1. *Under hypotheses* (6.17), (6.18), (6.19), (6.20), *and* (6.21), *if u is in $\mathcal{C}^2([0,T];W)$, then u_h satisfies the following error estimates.*

$$\|u - u_h\|_{\mathcal{C}^0([0,T];L)} \leq c_1 H^{k+1/2}, \tag{6.23}$$

$$\left[\frac{1}{T}\int_0^T \|u - u_h\|_V^2\right]^{1/2} \leq c_2 H^k, \tag{6.24}$$

where the constants c_1 and c_2 are bounded from above as follows.

$$c_1 \leq c\left[H + T\,(1+T)\right]^{1/2} \|u\|_{\mathcal{C}^2([0,T];W)}, \qquad c_2 \leq c\,[1+T]\,\|u\|_{\mathcal{C}^2([0,T];W)}.$$

Note that the norms used in the error estimates are the same as those of the stability estimates (6.14). The estimate (6.24) is optimal in the graph norm. The estimate (6.23) is the same as that obtained by the Discontinuous Galerkin technique (see for instance Johnson and Pitkäranta [34]).

The present theory can be extended to the coercive case, for instance in the case where the differential operator is of the form $A + \varepsilon D$, A being a first order differential operator and D a coercive second order differential operator. From a mathematical point of view, the coercivity of D implies that the evolution equation is parabolic. If ε is $\mathcal{O}(1)$, the Galerkin technique is optimal, but if ε is small, the coercivity is not strong enough to guarantee that the Galerkin approximation is satisfactory. It is shown in [26] that by using the same two-level framework as above and by perturbing the Galerkin technique with the same bilinear form b_h as above, Theorem 6.1 still holds. The remarkable result here is that the estimates are uniform with respect to ε, and optimal convergence is obtained in the graph norm of A. If we think of A as being a transport operator and D being a Laplacian, then convergence on the gradient of the solution in the streamwise direction is guaranteed whereas only L^2 convergence is guaranteed in the crosswind direction.

6.3.3. Examples and extension to the nonlinear case

A variety of discrete functional frameworks satisfying the hypotheses above are described in [25, 27, 26]. We choose to describe here the simplest one, *i.e.* the $\mathbb{P}_1$ framework, as it is probably used implicitly by many authors in LES. For the sake of simplicity we assume that Ω is a polyhedron in $\mathbb{R}^d$ ($d = 2$ or 3) and $\mathcal{T}_H = \cup\{K_H\}$

is a regular triangulation of Ω composed of affine simplices. Let us assume that V is composed of vector-valued functions in $\mathbb{R}^m$. Let us define first X_H by

$$X_H = \{v_H \in H^1(\Omega)^m;\ v_{H|K_H} \in \mathbb{P}_1(K_H)^m,\ \forall K_H \in \mathcal{T}_H\}. \tag{6.25}$$

In 2D, from each triangle $K_H \in \mathcal{T}_H$, we create four new triangles by connecting the middles of the three edges of K_H. In 3D, from each tetrahedron, we create eight new tetrahedra by proceeding as follows: on each face we connect the middles of the edges, and choosing arbitrarily two non-intersecting edges we connect the middles of these two edges. Let $h = H/2$ and $\mathcal{T}_h$ denote the resulting new triangulation. For each macro-simplex K_H, we define $\mathbb{P}$ as being the space of functions that are continuous on K_H, vanish at the vertices of K_H, and are piecewise $\mathbb{P}_1$ on each sub-simplex of K_H. We define

$$X_h^H = \{v_h^H \in H^1(\Omega)^m \mid v_{h|K_H}^H \in \mathbb{P}^m,\ \forall K_H \in \mathcal{T}_H\}. \tag{6.26}$$

By setting $X_h = X_H \oplus X_h^H$, it is clear that we can characterize X_h by

$$X_h = \{v_h \in H^1(\Omega)^m \mid v_{h|K_h} \in \mathbb{P}_1(K_h)^m,\ \forall K_h \in \mathcal{T}_h\}. \tag{6.27}$$

The couple (X_H, X_h) is referred to as the two-level $\mathbb{P}_1$ setting.

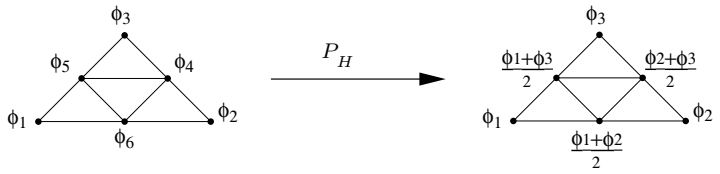

FIG. 4. Definition of P_H for the two-level $\mathbb{P}_1$ setting.

In Figure 4 we show a schematic representation of the action of the filter $P_H : X_h \longrightarrow X_H$ on a macro-element K_H of $\mathcal{T}_H$.

Numerical tests presented in [25, 27, 26] reveal that the present technique performs as predicted in the theory. However, when it is tested on rough solutions, *i.e.* discontinuous solutions or solutions exhibiting shocks, localized spurious oscillations are still present in the vicinity of discontinuities. These residual oscillations are due to the Gibbs phenomenon, which is well-known to those working on nonlinear conservation laws. It is the manifestation of a well-known theorem in analysis that states that truncated Fourier series of a given function does not converge uniformly to the function in question unless the function is very smooth (continuity is not enough), see Rudin [64, p. 97–98] for more details. A simple trick to eliminate this undesired oscillations consists of adding strong dissipation in the region where the solution is rough. Of course, one does not know *a priori* where the solution is rough, but one may expect that in this region the quantity $\nabla u_h^H = \nabla(u_h - P_H u_h)$ is of the same order as ∇u_h. Hence, we are led to introduce

the following shock-capturing nonlinear form:

$$c_h(u_h; u_h, v_h) = c_{sc} H \int_\Omega \frac{|\nabla u_h^H|}{|\nabla u_h|} \nabla u_h \cdot \nabla v_h \, \mathrm{d}x. \tag{6.28}$$

Then, (6.22) must be modified by replacing $b_h(u_h^H, v_h^H)$ by $b_h(u_h^H, v_h^H) + c_h(u_h^H, u_h, v_h)$. Unfortunately, no theory is available for supporting the presence of c_h, though it has been shown to be extremely efficient in applications. Note that in regions where $|\nabla u_h^H|$ is of order $|\nabla u_h|$, the added viscosity is $\mathcal{O}(H)$ as it should be. Note also the similarity between this term and (6.2) and (6.3) where for dimensional reasons $H|\nabla u_h^H|/|\nabla u_h|$ is replaced by $H^2|\nabla u_h^H|$; hence, we can also interpret (6.4) as a shock-capturing operator.

We also mention that if in (6.28), $|\nabla u_h^H|$ is replaced by the residual of the equation, then it can be proved for scalar conservation laws that this term yields an L^∞-estimate on the approximate unknown (*i.e.* a maximum principle) that guarantees convergence to the entropy solution (see the series of papers by Johnson, Szepessy *et al.* [35, 36, 71]).

We now provide a spectral interpretation of the respective effects of b_h and c_h. The bilinear form b_h is a viscosity term that takes care of the non-coercive (*i.e.* hyperbolic) character of the equation, whereas c_h is a shock capturing term that takes care of the Gibbs phenomenon induced by discontinuities. The bilinear form b_h suppresses wide-spreading (*i.e. unlocalized*) node-to-node oscillations induced by the lack of coercivity. As a result, this term attenuates only the high wave-number modes, possibly producing a dip on the tail of the spectrum of the solution as observed in [7] when Galerkin Least-Squares is used alone to simulate the Kolmogorov cascade. On the other hand, Gibbs oscillations induced by discontinuities are very localized around the discontinuities, which means that their spectral range is wide and centered in the intermediate wave-number modes. As a result, c_h acts on intermediate wave-number modes. These crude observations may explain why stabilization and shock capturing must be used jointly to reproduce the $k^{-5/3}$ cascade in LES as observed in [7]. In conclusion, the selection of the values of the constants scaling b_h and c_h is an issue that seems to be important in practice. Some tentative mathematical answers such as in [5] could be proposed to evaluate optimal constants, but also a pragmatic option could be to utilize one of the strategies suggested by Lilly in [52].

7. Concluding remarks

The motivation and objectives in writing the present paper were primarily to advance mathematical arguments to justify LES modeling. We have analyzed filtering techniques, nonlinear viscosity methods, spectral eddy-viscosity methods, and two-scale methods. The analysis of these different LES strategies has revealed mathematical patterns that suggest criteria for developing a rigorous mathematical theory for LES. Among the possible criteria, we propose the following:

(i) **Regularization:** LES should regularize the Navier–Stokes equations; that is, LES should transform the (possibly ill-posed) Navier–Stokes equations into a well-posed set of PDE's.

(ii) **Selection of a physically relevant solution:** LES should select physically relevant solutions of the Navier–Stokes equations; that is, LES should select "dissipative solutions" *à la* Duchon–Robert [13, 63] or "suitable weak solutions" *à la* Caffarelli, Khon, Nirenberg [6], and Scheffer [67].

The filtering techniques, such as the Leray regularization and the NS-α model, satisfy these two criteria. These techniques are based on a partial filtering of the nonlinear advection term. We believe they are the most appropriate justifications for the filtering methodologies currently employed in LES. In passing, we have shown that filtering the NS equations and closing them as accurately as possible, *i.e.* possibly exactly, in the hope that the resulting problem is simpler to solve, is a paradoxical program. Indeed, as shown by Germano [20, 21], there exist filters for which exact closure is possible. For such filters, the solution sets of the filtered and unfiltered Navier–Stokes equations are isomorphic; as a result, no gain should be expected since the number of degrees of freedom required to represent the filtered and the unfiltered solutions are identical.

The nonlinear viscosity methods like those proposed by Kaniel [37], Ladyženskaja [43, 42], and Smagorinsky [69] satisfy both criteria. These techniques consist in adding a vanishing nonlinear viscosity to the momentum equation. Contrary to what is frequently claimed in the literature, these regularization techniques do not require any kind of filtering to be justified.

We have studied vanishing eddy-viscosity methods based on spectral approximation. Comparing Kraichnan-like eddy-viscosity and Tadmor's spectral viscosity, it appears that the former pollutes the spectral accuracy whereas the latter is optimal in this respect. Contrary to what is sometimes claimed, we have shown that these methods are radically different from the nonlinear regularizations of Kaniel, Ladyženskaja, and Smagorinsky. Nonlinear regularizations yield unique solutions, whereas uniqueness is an open question for vanishing eddy-viscosity and spectral viscosity methods. Hence, spectral eddy-viscosity methods do not comply with the first criterion we have proposed. As a remedy, we have suggested a model with a spectral hyper-viscosity that preserves spectral accuracy and complies with criteria (i) and (ii).

Finally, we have reviewed two-scale LES approximation techniques. The analysis of existing techniques has led us to propose a new dispersive scale-similarity model that satisfies an energy inequality. On the other hand, we have been unable to derive a rigorous mathematical result in favor of these methods. The best justification we can bring forward is that two-scale LES methods are related to a subgrid stabilization technique that solves non-coercive PDE's. However, we are confident that further mathematical research on this subject will bring new insights for LES modeling.

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