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SPECTRAL VANISHING VISCOSITY METHOD FOR HIGH-ORDER LES : COMPUTATION OF THE DISSIPATION RATES

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Abstract. *The spectral vanishing viscosity (SVV) method has recently appeared of interest for the large-eddy simulation (LES) of turbulent flows. Here we show that the SVV-LES approach allows to easily compute turbulent quantities, like the dissipation rates of the turbulent kinetic energy or of the variance of a passive scalar and to clearly discern the subgrid and viscous contributions. Results obtained for the turbulent wake of a cylinder in a weakly stratified fluid are provided.*

1 INTRODUCTION

The Spectral Vanishing Viscosity (SVV) method, first developed to handle 1D hyperbolic problems with spectral methods [14, 22], has recently appeared of interest for the large-eddy simulation (LES) of turbulent, and so multiscale, flows [8, 15]. Basically, the idea is to stabilize the computation by completing the momentum equation with a dissipation term only active in the high-frequency range of the spectral approximation.

The main advantage of the SVV method is to preserve the exponential rate of convergence of the numerical solution towards the exact (smooth) one. Moreover, numerical experiments have shown that the flows thus computed actually show the characteristics of turbulent flows, in terms of statistical moments and agreement with Kolmogorov theory. This has given rise to a SVV-LES approach for the computation of turbulent flows [9, 16, 17, 18].

Here our goal is to show that in the frame of a SVV-LES it is easy to provide a detailed analysis of the dissipation of the energy, say of the transfer of energy from the resolved to the non-resolved scales. This results from the fact that, contrarily to some other Implicit LES (ILES) or MILES (Monotone Integrated LES) approaches [1, 21], just like in classical LES the stabilization term is explicitly introduced. With respect to a low-order classical

LES, using a high-order method shows the advantage that there is no mixing between the approximation and modeling errors, so that one can expect to obtain accurate estimations of turbulent quantities, like the dissipation rates of the turbulent kinetic energy or of the variance of a passive scalar.

Note that the SVV-LES approach is close to multiscale formulations, which make use of artificial dissipation at the smallest scales to stabilize Galerkin approximations of transport dominated problems [5, 6].

Our plan is the following : In Section 2 we recall how to express the SVV term in a multidimensional framework and then express the dissipation rate and pseudo-dissipation rate of the turbulent kinetic energy. A similar approach is used to express the dissipation rate of the variance of a passive scalar. In Section 3 we provide results obtained for the wake of a cylinder in a thermally stratified fluid. We conclude in Section 4.

2 DISSIPATION RATES IN THE SVV-LES FORMULATION

2.1 The SVV stabilization term

We are interested in the stabilization of the spectral approximation of some transport-diffusion problem. With N for the space discretization parameter, in the 1D scalar case the SVV stabilization term reads :

$$V_N = \epsilon_N \partial_x (Q_N(\partial_x u_N)) \tag{1}$$

where ϵ_N is a $O(1/N)$ coefficient, Q_N a spectral viscosity operator and $u_N(x,t)$ the numerical solution (x and t are the space and time variables). Thus, in the non-periodic case investigated in [14], with $u = \sum_{k\geq 0} \hat{u}_k L_k \in L^2(-1,1)$, where L_k is the Legendre polynomial of degree k :

$$Q_N(u) \equiv \sum_{k=0}^{N} \hat{Q}_k \hat{u}_k L_k \tag{2}$$

where $\hat{Q}_k = 0$ if $k \leq m_N$ and $0 < \hat{Q}_k < \hat{Q}_{k+1} \leq 1$ if $k > m_N$, with e.g. $m_N = \sqrt{N}$ and $\hat{Q}_k = \exp(-(k-N)^2/(k-m_N)^2)$.

The extension to the multidimensional case is not trivial, as shown by the different variants met in the literature [4, 7, 8, 23]. For us we advocate the following form :

$$V_N = \nabla \cdot (\epsilon_N Q_N(\nabla u_N)), \quad \epsilon_N Q_N \equiv diag\{\epsilon_{N_i} Q^i_{N_i}\} \tag{3}$$

with $Q^i_{N_i}$ the 1D SVV operator acting in direction x_i. Thus, we actually use a diagonal matrix form of the operator $\epsilon_N Q_N$, just as one introduces a non-scalar diffusivity when anisotropic media are considered. Moreover, if a mapping $\boldsymbol{f} : \hat{\Omega} \rightarrow \Omega$ from the reference domain $\hat{\Omega}$ to the physical domain Ω is involved, then we use :

$$\epsilon_N Q_N(\nabla u_N) \equiv \epsilon_N Q_N(\hat{\nabla}(u_N \circ \boldsymbol{f}))\,(G \circ \boldsymbol{f}) \tag{4}$$

where G is the Jacobian matrix of $\boldsymbol{f}^{-1}$ and where $\hat{.}$ refers to the variables of the reference domain.

The SVV stabilization can be simply implemented by combining the diffusion and SVV terms to obtain, with $\lambda > 0$ for a conductivity coefficient :

$$\nabla \cdot (\lambda \nabla u_N) + V_N = \nabla \cdot \lambda S_N(\nabla u_N) \tag{5}$$

where $S_N \equiv 1 + \lambda^{-1} \epsilon_N Q_N$. With our definition of $\epsilon_N Q_N$ and with ∂_j for the differentiation with respect to x_j, we then have :

$$\nabla \cdot (\lambda \nabla u_N) + V_N = \sum_j \partial_j \left(\lambda \, \tilde{\partial}_j u_N \right) \tag{6}$$

where $\tilde{\partial}_j = (1 + \lambda^{-1} \epsilon_{N_j} Q^j_{N_j}) \partial_j$.

We are now ready to express the dissipation rates of the turbulent kinetic energy and of the variance of a passive scalar for an incompressible flow. These quantities appear as sink terms in the corresponding evolution equations, which are derived from the incompressible Navier-Stokes and transport-diffusion equations, respectively.

2.2 Dissipation rate of the turbulent kinetic energy

The approach follows what is usually done, see e.g. [20] for details, except that we start from the SVV-stabilized incompressible Navier-Stokes equations :

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i = -\partial_i \bar{p} + \nu \partial_j \tilde{\partial}_j \bar{u}_i \tag{7}$$

$$\partial_j \bar{u}_j = 0 \tag{8}$$

where summation over repeated indices is assumed, with $\tilde{\partial}_j$ based on the kinematic viscosity ν and where $\bar{u}_i$ and $\bar{p}$ stand for the SVV-LES approximations of the i-component of the velocity and of the pressure divided by the density, respectively. Note that we use the usual LES notations, with an over-bar to express the fact that at the end we will only compute some " filtered quantities", even if we have not followed the usual LES methodology, based on the filtered Navier-Stokes equations.

To obtain an equation for the specific kinetic energy of the filtered velocity, $e_f = \bar{u}_i \bar{u}_i / 2$, we multiply eq. (7) by $\bar{\boldsymbol{u}}$ to obtain :

$$\bar{D}_t e_f + \partial_j (\bar{u}_j \bar{p} - \nu \bar{u}_i \tilde{\partial}_j \bar{u}_i) = -\tilde{\epsilon} \tag{9}$$

with $\bar{D}_t \equiv \partial_t + \bar{u}_j \partial_j$ and where appears the pseudo-dissipation term

$$\tilde{\epsilon} = \nu \partial_j \bar{u}_i \tilde{\partial}_j \bar{u}_i \, . \tag{10}$$

Here the flux term does not show the work of the viscous and sub-grid forces, which should be associated for an incompressible Newtonian flow to the tensor : $\tilde{S}_{ij} = (\tilde{\partial}_i \bar{u}_j + \tilde{\partial}_j \bar{u}_i)/2$. With a correct formulation of the flux one has :

$$\bar{D}_t e_f + \partial_j (\bar{u}_j \bar{p} - 2\nu \bar{u}_i \tilde{S}_{ij}) = -\epsilon$$

where appears the dissipation term :

$$\epsilon = \nu(\partial_j \bar{u}_i \tilde{\partial}_j \bar{u}_i + \partial_j(\bar{u}_i \tilde{\partial}_i \bar{u}_j)) \,. \tag{11}$$

If we assume that ∂_j and $\tilde{\partial}_j$ commute, with the continuity equation and thanks to the symmetry of the $\tilde{S}_{ij}$ tensor, one obtains :

$$\epsilon = 2\nu S_{ij} \tilde{S}_{ij} \,. \tag{12}$$

Without SVV stabilization, then $\tilde{\partial}_j \equiv \partial_j$ and $\tilde{S}_{ij} \equiv S_{ij}$, so that one recovers the usual form of the dissipation term. Note however that $S_{ij} S_{ij} \geq 0$, whereas ϵ, from eq. (12), may be negative, giving then the possibility of a local transfer of energy from the non-resolved to the resolved scales (backscatter phenomenon).

From the eq. (10) and (12), it is now possible to compute the pseudo-dissipation and dissipation rates of the turbulent kinetic energy. Thus, for the dissipation rate, with $< . >$ to denote a statistical mean,

$$< \epsilon >= 2\nu < S_{ij} >< \tilde{S}_{ij} > + \varepsilon$$

where the first term in the right hand side is the dissipation due to the mean flow and ε the dissipation rate of the turbulent kinetic energy :

$$\varepsilon = 2\nu(< S_{ij} \tilde{S}_{ij} > - < S_{ij} >< \tilde{S}_{ij} >) \,. \tag{13}$$

Similarly, the pseudo-dissipation rate writes :

$$\tilde{\varepsilon} = \nu(< \partial_j \bar{u}_i \tilde{\partial}_j \bar{u}_i > - < \partial_j \bar{u}_i >< \tilde{\partial}_j \bar{u}_i >) \,. \tag{14}$$

Remark : To obtain the expression (12) we have assumed that the operators ∂_j and $\tilde{\partial}_j$ commute. This is exact in the Fourier case, since for the mode $\boldsymbol{k}$, $\partial_j = \mathrm{i}k_j$ and $\tilde{\partial}_j \propto \mathrm{i}k_j$. There is however an approximation in the Legendre or Chebyshev cases.

2.3 Dissipation rate for a passive scalar

The approach is quite similar. With $\tilde{\partial}_j$ based on the diffusivity a of the passive scalar T, the evolution equation reads :

$$\partial_t \bar{T} + \bar{u}_j \partial_j \bar{T} = a \partial_j \tilde{\partial}_j \bar{T}$$

so that by multiplying by $\bar{T}$ and with $e_T = \bar{T}\bar{T}$ we obtain :

$$\begin{aligned} \bar{D}_t e_T &= 2a\bar{T}\partial_j \tilde{\partial}_j \bar{T} \\ \bar{D}_t e_T + \partial_j(-2a\bar{T}\tilde{\partial}_j \bar{T}) &= -\epsilon_T \end{aligned} \tag{15}$$

with the dissipation term :

$$\epsilon_T = 2a\partial_j \bar{T} \tilde{\partial}_j \bar{T} \,. \tag{16}$$

For the dissipation rate of the variance of T this yields :

$$\varepsilon_T = 2a(< \partial_j \bar{T} \tilde{\partial}_j \bar{T} > - < \partial_j \bar{T} >< \tilde{\partial}_j \bar{T} >) \,. \tag{17}$$

3 Application to the stratified wake of a cylinder

Here we focus on the turbulent wake of a cylinder in a weakly linearly stratified fluid. The Reynolds number equals $Re = 3900$ and the Prandtl number $Pr = 7$. The Bousssinesq approximation is used to model the thermal-hydrodynamic, with an internal Froude number $F = U/ND = 75$, where U is the upstream velocity, N the Brunt-Väisälä angular frequency and D the diameter of the cylinder. For the non-stratified case, which constitutes a standard test-case [10, 12], comparisons with experimental results are provided in [18] and a sensitivity study to the SVV parameters is carried out in [17].

3.1 Characteristics of the computation

The main characteristics of the computer code are the following :

- The time-scheme is second-order accurate and makes use of three steps : (i) Transport step : The convective terms are handled explicitly with an OIF (Operator Integration Factor) semi-Lagrangian method [13]. (ii) Diffusion step : the diffusion terms are handled implicitly. (iii) Projection step : a Darcy problem is solved to obtain a divergence free velocity field and the pressure is then updated.
- A multi-domain technique is used in the elongated streamwise direction. In each subdomain a Chebyshev collocation method is used in the streamwise and crossflow directions, whereas a Fourier Galerkin method is used in the homogeneous spanwise direction.
- The obstacle is modeled by using a " pseudo-penalization" technique.
- The code is vectorized and parallelized, each subdomain being associated to a different processor.

Details on the algorithm may be found in [3, 16, 19].

The characteristic parameters of the computation are :

- Computational domain : $\Omega = (-6.5, 17.5) \times (-4, 4) \times (-2, 2)$. The cylinder is of unit diameter and centered at $x = y = 0$ (x, y, z : streamwise, crossflow vertical and spanwise directions).
- Initial conditions : the fluid is at rest and linearly stratified : $T = y$ in dimensionless form. Boundary conditions : free-slip and adiabaticity conditions at $y = \pm 4$, Dirichlet conditions, $\boldsymbol{u} = (1, 0, 0)$ and $T = y$, at $x = -6.5$ (inlet), advection at the mean flow velocity at $x = 17.5$ (outlet).
- Mesh : Number of subdomains : 5, the interfaces of the subdomains are located at $x = \{-0.5, 2.5, 6.5, 11.5\}$; Polynomial approximation degrees in each subdomain : $N_1 = 60$, $N_2 = 120$ in x and y directions, respectively ; Number of Fourier grid points : $N_F = 60$. Time step : $5.\,10^{-3}$.
- SVV parameters : $m_N = N/2$ and $\epsilon_N = 1/N$.
- The statistics have been computed on 150 dimensionless time units (D/U).

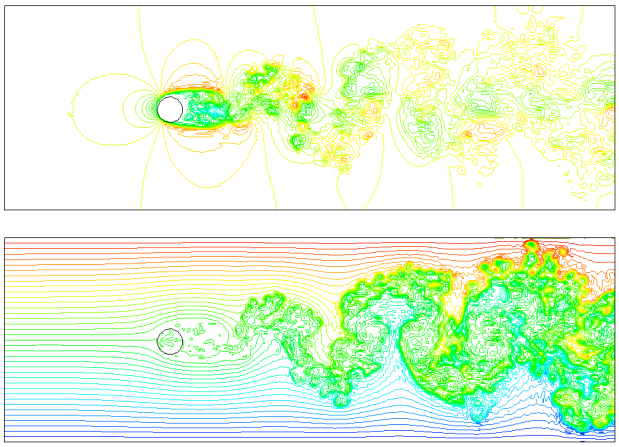

FIG. 1: Streamwise velocity (top) and temperature (bottom) instantaneous fields in the median vertical plane, $Re = 3900$, $F = 75$, $Pr = 7$.

3.2 Numerical results

Qualitative results on the computed flow are shown in Fig. 1, where instantaneous temperature and streamwise velocity fields are visualized. Such results are very similar to those obtained when the stratification is not active ($F = \infty$), in agreement with experimental investigations [2]. In a referential moving with the fluid, one expects, in the high Foude number regime, to have an influence of the stratification for $Nt/(2\pi) \geq 0.1$, where t stands here for an elapsed time [11]. With $F = U/ND$, one may then expect to have an influence of the stratification for $x \geq 0.2\pi F \approx 47 \gg 17.5$. More quantitatively, we have checked that for $x \leq 17.5$ the flows obtained with $F = 75$ and $F = \infty$ are similar in terms of statistical moments or power spectra. Here we rather focus on the turbulent kinetic energy, temperature variance and corresponding dissipation rates.

In Fig. 2 (left) are shown profiles of the turbulent kinetic energy at different streamwise locations (top) and the x-variations along the streamwise centerline $y = z = 0$ (bottom). Similar results for the temperature variance are shown in Fig. 2 (right). Such results are not completely statistically converged, with a few per cents variations in the homogeneous spanwise direction and some symmetry deficiencies in the profiles. Clearly the behaviors of the turbulent kinetic energy and temperature variance are strongly different. The former shows a maximum for $x \approx 2.5$ and then smoothly decays whereas the latter goes on in increasing at the outlet of the computational domain. This is associated to the fact that the characteristic time-scale associated to thermal diffusion, i.e. D^2/a, where a is here the thermal diffusivity, is much greater than the inertial time scale D/U. Moreover, the temperature variance does not correspond to a physical energy.

The dissipation rates of the turbulent kinetic energy, times the Reynolds number $Re = 3900$, and temperature variance, times the Péclet number $Pe = 27300$, are shown in Fig. 3. Such results have been obtained from the eq. (13) and (17), with $\nu = a = 1$.

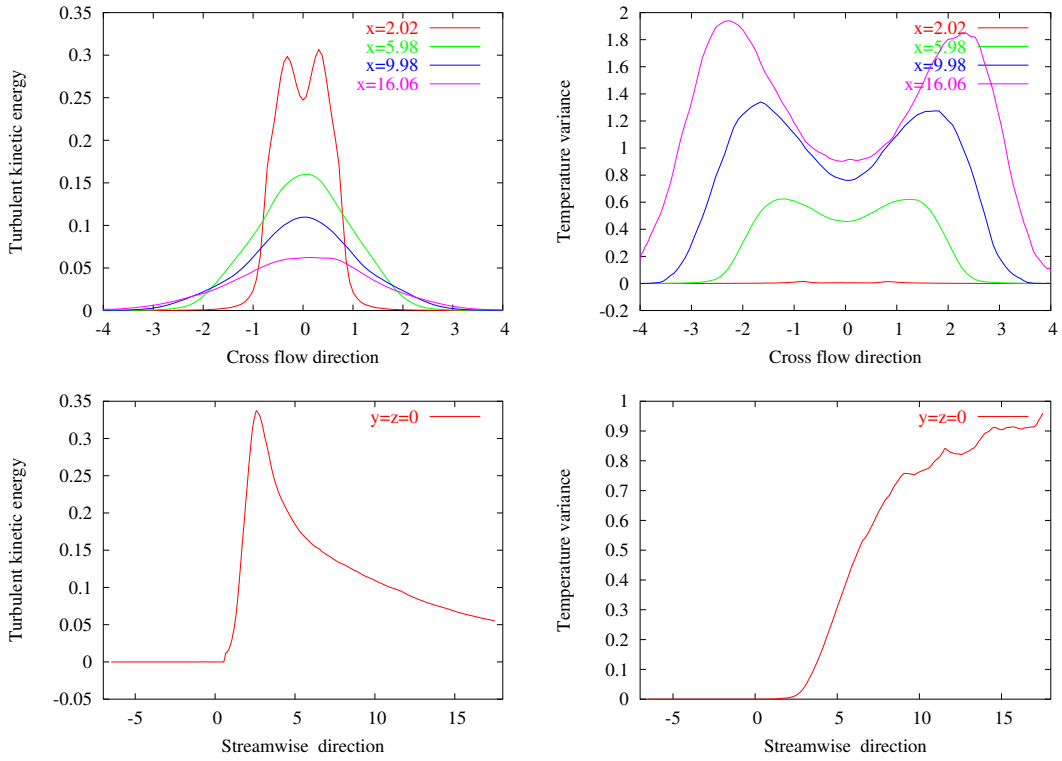

FIG. 2: Turbulent kinetic energy (left) and temperature variance (right) at different x-locations (top) and along the streamwise central line (bottom).

The thermal profiles are especially non-smooth, with sharp variations in the streamwise direction. These sharp variations are partially associated to the subdomain decomposition, but the general trend is captured. Here again one observes very different behaviors for the turbulent kinetic energy, with a maximum of the dissipation rate at $x \approx 2$, and for the temperature variance.

Finally, we compare in Fig. 4 the contributions of the SVV stabilization term and of the diffusive term by plotting the dissipation rates, from the eq. (13) and (17), and those obtained when using ∂_j rather than the SVV modified derivative $\tilde{\partial}_j$. For the turbulent kinetic energy the contributions are roughly of same magnitude, whereas for the temperature variance the stabilization term is dominant, essentially because the Péclet number is much larger than the Reynolds number ($Pe/Re = Pr = 7$). For the turbulent kinetic energy we also compare the dissipation and pseudo-dissipation rates, as computed with the eq. (13) and (14), respectively. As usually found with the classical definitions of these quantities, i.e. with $\tilde{\partial}_j = \partial_j$, the two expressions give very close results.

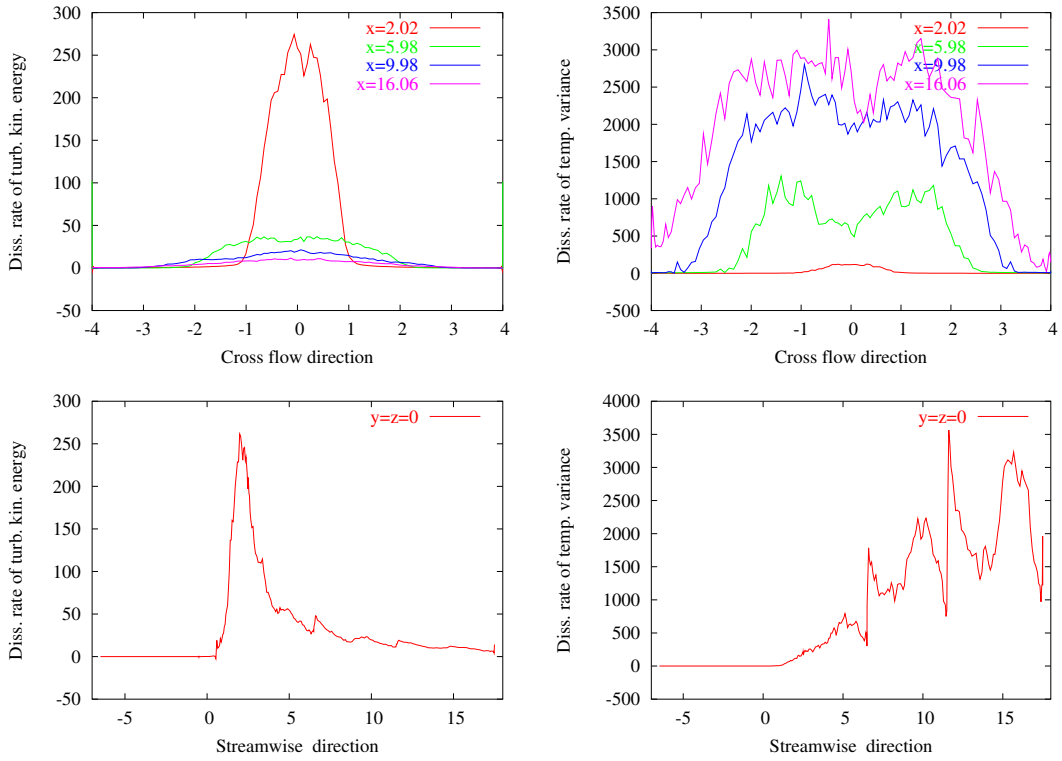

FIG. 3: Dissipation rates of the turbulent kinetic energy times Re (left) and temperature variance times Pe (right) at different x-locations (top) and along the streamwise central line (bottom).

4 CONCLUSION

For a good understanding of turbulent flows a detailed analysis of turbulence quantities, like the turbulent kinetic energy and dissipation rate of this energy, is required. There is generally no problem to compute the turbulent kinetic energy from the filtered velocity, because the contribution of the non-resolved small scales is negligible. However, this no-longer holds for the dissipation rate, because the energy transfer occurs essentially from the resolved to the non-resolved scales. Here we have shown that just for a classical LES, i.e. based on a modeling of the "sub grid stress tensor", it is easy in the SVV-LES framework to express the dissipation rate of the turbulent kinetic energy and to clearly discern the diffusive and subgrid parts. This is generally not true for ILES methods and with respect to a low-order classical LES, using a high-order approach allows us to eliminate the numerical dissipation. A similar work has been carried out for a passive scalar and results obtained for the turbulent wake of a cylinder in a weakly stratified fluid have been presented.

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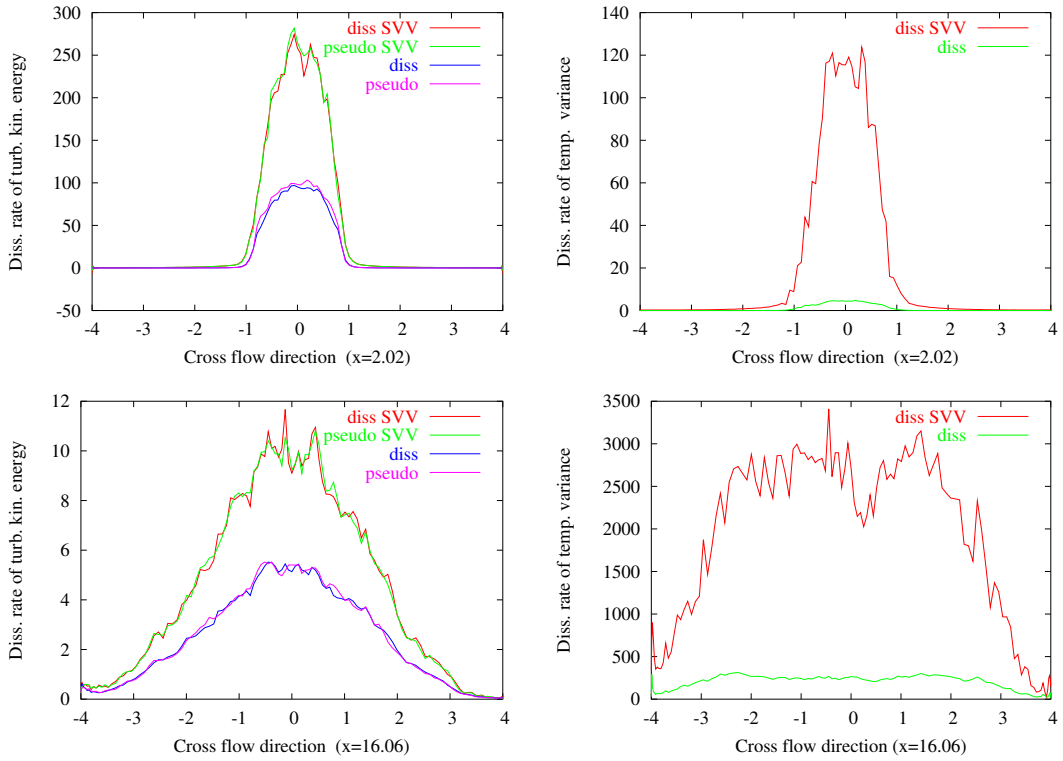

FIG. 4: Dissipation and pseudo-dissipation rates of the turbulent kinetic energy times Re (left) and dissipation rate of the temperature variance times Pe (right) from the SVV modified and standard formulations of these rates at two x-locations.

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RÉFÉRENCES

[1] J.P. Boris, F.F. Grinstein, E.S. Oran and R.L. Kobe, New insights into large eddy simulation, *Fluid Dyn. Res.*, **10**, 199-228 (1992).

[2] D.L. Boyer, P.A. Davies, H.J.S. Fernando and X. Zhang, Linearly stratified flow past a horizontal cylinder, *Phil. Trans. R. Soc. Lond.*, **A 328**, 501-528 (1989).

[3] L. Cousin and R. Pasquetti, High-order methods for the simulation of transitional to turbulent wakes, Advances in Scientific Computing and Applications, Editors Y. Lu, W. Sun, T. Tang, Sciences Press Beijing/New York, 133-143 (2004).

[4] G.Q. Chen, Q. Du and E. Tadmor, Spectral viscosity approximations to multidimensional scalar conservations laws, *Math. of Comput.*, **204**, 629-643 (1993).

[5] J.L. Guermond, Stabilization of Galerkin approximations of transport equations by subgrid modeling, *M2AN*, **33** (6), 1293-1316 (1999).

[6] T.J.R. Hugues, L. Mazzei and A.A. Oberai, The multiscale formulation of large eddy simulation : Decay of homogeneous isotropic turbulence, *Phys. Fluids*, **13** (2), 505-512 (2001).

[7] O.S.M. Kaber, A Legendre pseudo-spectral viscosity method, *J. Comput. Phys.*, **128**(1), 165-180 (1995).

[8] G.S. Karamanos and G.E. Karniadakis, A spectral vanishing viscosity method for large-eddy simulation, *J. Comput. Phys.*, **163**, 22-50 (2000).

[9] R. Kirby and G.E. Karniadakis, Coarse resolution turbulence simulations with spectral vanishing viscosity-large eddy simulations, *J. Fluid Engng.*, **124**, 886-891 (2002).

[10] A.G. Kravchenko and P. Moin, Numerical studies of flow over a circular cylinder at Re=3900, *Phys.Fluids*, **12** (2), 403-417 (2000).

[11] J.T. Lin and Y.H. Pao, Wakes in stratified fluids, *Ann. Rev. Fluid Mech.*, **11**, 317-338 (1979).

[12] X. Ma, G.S. Karamanos and G.E. Karniadakis, Dynamic and low-dimensionality of a turbulent near wake, *J. Fluid Mech.*, **410**, 29-65 (2000).

[13] Y. Maday, A.T. Patera and E.M. Ronquist, An operator-integration-factor splitting method for time-dependent problems : application to incompressible fluid flow, *J. of Sci. Comp.*, **5** (4), 263-292 (1990).

[14] Y. Maday, S.M.O. Kaber and E. Tadmor, Legendre pseudo-spectral viscosity method for nonlinear conservation laws, *SIAM J. Numer. Anal.*, **30** (2), 321-342 (1993).

[15] R. Pasquetti and C.J. Xu, High-order algorithms for large eddy simulation of incompressible flows, *J. Sci. Comput.*, **17** (1-4), 273-284 (2002).

[16] R. Pasquetti, High-order LES modeling of turbulent incompressible flows, *C.R. Acad. Sci. Paris*, **333**, 39-49 (2005).

[17] R. Pasquetti, Spectral vanishing viscosity method for LES : Sensitivity to the SVV control parameters, *J. of Turbulence*, **vol. 6, N12**, Special issue : Marseille Euromech Colloquium (2005).

[18] R. Pasquetti, Spectral vanishing viscosity method for large-eddy simulation of turbulent flows, *J. Sci. Comp.*, proc. of ICOSAHOM 2004 Congress (in press).

[19] R. Pasquetti, R. Bwemba and L. Cousin, A pseudo-penalization method for high Reynolds number unsteady flows, *Applied Numerical Mathematics*, accepted.

[20] S.B. Pope, *Turbulent flows*, Cambridge University Press (2000).

[21] P. Sagaut, *Large-eddy simulation for incompressible flows : An introduction*, Series : Scientific Computation, Springer (2005).

[22] E. Tadmor, Convergence of spectral methods for nonlinear conservation laws, *SIAM J. Numer. Anal.*, **26** (1), 30-44 (1989).

[23] C.J. Xu and R. Pasquetti, Stabilized spectral element computations of high Reynolds number incompressible flows, *J. Comput. Phys.*, Vol 196/2 pp. 680-704 (2004).