Problem Sheet 2 (Do Carmo: Chapter 3, Chapter 1)

Differentiable Manifolds, Calculus of Differential Forms

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I. Examples of Differentiable Manifolds

1. Real projective space. Define real projective n-space by

$$\mathbb{RP}^n = \big(\mathbb{R}^{n+1} \setminus \{0\}\big)/\!\sim,$$

where

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n), \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

Write $[x_0 : \cdots : x_n]$ for the equivalence class of $(x_0, \dots, x_n)$. For $i = 0, \dots, n$, let

$$V_i = \big\{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\big\}.$$

Define $\varphi_i : V_i \to \mathbb{R}^n$ by dividing all coordinates by x_i and omitting the ith coordinate.

(a) Prove that φ_i is well defined and bijective, and write down φ_i^{-1} explicitly.

(b) Prove that the sets $V_0, \dots, V_n$ cover $\mathbb{RP}^n$.

(c) For $i \neq j$, determine the open set $\varphi_i(V_i \cap V_j) \subseteq \mathbb{R}^n$ and compute the transition map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \longrightarrow \varphi_j(V_i \cap V_j).$$

Prove that this transition map is smooth.

(d) Conclude that the charts (V_i, φ_i) give $\mathbb{RP}^n$ the structure of a differentiable n-manifold.

(e) Show that $\mathbb{RP}^1$ is diffeomorphic to the circle S^1.

2. The product of two differentiable manifolds. Let M^m and N^n be differentiable manifolds. Suppose that

$$\{(U_\alpha, \varphi_\alpha)\} \qquad \text{and} \qquad \{(V_\beta, \psi_\beta)\}$$

are differentiable atlases for M and N, respectively. For each pair (α, β), define

$$\Phi_{\alpha\beta} : U_\alpha \times V_\beta \longrightarrow \mathbb{R}^{m+n}$$

by

$$\Phi_{\alpha\beta}(p, q) = \big(\varphi_\alpha(p), \psi_\beta(q)\big).$$

(a) Prove that the sets $U_\alpha \times V_\beta$ cover $M \times N$ and that each $\Phi_{\alpha\beta}$ is a homeomorphism onto an open subset of $\mathbb{R}^{m+n}$.

(b) Compute the transition map between two overlapping product charts and prove that it is smooth.

(c) Deduce that $M \times N$ has a natural structure of a differentiable manifold of dimension $m + n$.

(d) Prove that the coordinate projections

$$\pi_M : M \times N \to M, \qquad \pi_N : M \times N \to N$$

are smooth.

II. Foundations of manifolds

Two Definitions of a Smooth Manifold

Do Carmo's definition. An n-dimensional smooth manifold consists of a set M together with a maximal family of injective maps

$$f_\alpha : \Omega_\alpha \longrightarrow M, \qquad \Omega_\alpha \subseteq \mathbb{R}^n \text{ open},$$

called *parametrizations*, satisfying the following conditions:

(i) the sets $U_\alpha = f_\alpha(\Omega_\alpha)$ cover M;

(ii) whenever $U_\alpha \cap U_\beta \neq \varnothing$, the sets

$$f_\alpha^{-1}(U_\alpha \cap U_\beta) \quad \text{and} \quad f_\beta^{-1}(U_\alpha \cap U_\beta)$$

are open in $\mathbb{R}^n$, and the change-of-parameters maps $f_\beta^{-1} \circ f_\alpha$ and $f_\alpha^{-1} \circ f_\beta$ are smooth on their respective domains.

This family induces a topology on M: a subset $A \subseteq M$ is declared open if

$$f_\alpha^{-1}(A \cap U_\alpha)$$

is open in Ω_α for every α. Following Do Carmo, we assume that this topology is Hausdorff and has a countable basis.

Topology-first definition. In class, we said that an n-dimensional smooth manifold is a Hausdorff, second-countable topological space M together with a maximal family of charts

$$\varphi_\alpha : U_\alpha \longrightarrow \Omega_\alpha \subseteq \mathbb{R}^n,$$

such that the sets U_α are open and cover M, each φ_α is a homeomorphism onto an open subset of $\mathbb{R}^n$, and all transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1}$$

are smooth wherever they are defined.

The next problem shows that these two definitions are equivalent.

3. Equivalence of the two definitions.

(a) Begin with a manifold in Do Carmo's sense. Let $\mathcal{T}$ be the collection of subsets $A \subseteq M$ such that

$$f_\alpha^{-1}(A \cap U_\alpha)$$

is open in Ω_α for every α.

Prove that $\mathcal{T}$ is a topology on M. Prove also that every U_α is open and that

$$f_\alpha : \Omega_\alpha \longrightarrow U_\alpha$$

is a homeomorphism. Conclude that

$$\varphi_\alpha = f_\alpha^{-1} : U_\alpha \longrightarrow \Omega_\alpha$$

defines a smooth atlas on $(M, \mathcal{T})$, hence Do Carmo's definition implies the one we used in class.

(b) Conversely, begin with a manifold in the topology-first sense and a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$. Define

$$\Omega_\alpha = \varphi_\alpha(U_\alpha), \qquad f_\alpha = \varphi_\alpha^{-1} : \Omega_\alpha \longrightarrow M.$$

Prove that the maps f_α satisfy Do Carmo's covering, openness, and change-of-parameters conditions.

4. Second countability and countable atlases. Suppose that M is Hausdorff and locally Euclidean, where second countability is not assumed. Prove that the topology of M is second-countable if and only if M admits a countable atlas of parametrizations

$$f_k : \Omega_k \longrightarrow f_k(\Omega_k) \subset M, \qquad k \in \mathbb{N},$$

where each $\Omega_k \subseteq \mathbb{R}^n$ is open.

III. Calculus of differential forms

5. Exterior products and exterior derivatives. On $\mathbb{R}^3$, with coordinates (x, y, z), consider the 1-forms

$$\alpha = x\,dy - y\,dx + z\,dz, \qquad \beta = z\,dx + x\,dz.$$

(a) Compute $\alpha \wedge \beta$ and write the answer in the ordered basis

$$dx \wedge dy, \qquad dx \wedge dz, \qquad dy \wedge dz.$$

(b) Compute $d\alpha$ and $d\beta$.

(c) Compute $d(\alpha \wedge \beta)$ directly from your answer in part (a).

(d) Verify explicitly the graded Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta.$$

6. Pulling back differential forms. Let

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad F(u, v) = \big(u \cos v, u \sin v, v\big),$$

and let (x, y, z) denote the standard coordinates on $\mathbb{R}^3$. Consider

$$\omega = x\,dy - y\,dx + dz, \qquad \eta = dx \wedge dy + dz \wedge dx.$$

(a) Compute $F^*(dx)$, $F^*(dy)$, and $F^*(dz)$.

(b) Compute and simplify $F^*\omega$ and $F^*\eta$.

(c) Compute $d\omega$ and verify directly that

$$d(F^*\omega) = F^*(d\omega).$$

(d) Explain why $F^*\tau = 0$ for every 3-form τ on $\mathbb{R}^3$.

7. Powers of the standard 2-form. On $\mathbb{R}^{2n}$ define

$$\Omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

For $k \geq 1$, write

$$\Omega^k = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{k \text{ factors}}.$$

(a) Prove that if γ is a differential form of odd degree, then $\gamma \wedge \gamma = 0$.

(b) Prove that, for $1 \leq k \leq n$,

$$\Omega^k = k! \sum_{1 \leq j_1 < \cdots < j_k \leq n} \big(dx_{2j_1-1} \wedge dx_{2j_1}\big) \wedge \cdots \wedge \big(dx_{2j_k-1} \wedge dx_{2j_k}\big).$$

(c) Deduce that

$$\Omega^n = n!\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}$$

and that $\Omega^{n+1} = 0$.

8. Jacobian determinants and volume forms. Let $U \subseteq \mathbb{R}^n$ be open and let

$$G : U \longrightarrow \mathbb{R}^n, \qquad G = (G^1, \ldots, G^n),$$

be smooth. Denote the standard coordinates on the target by $(y_1, \ldots, y_n)$ and those on the domain by $(x_1, \ldots, x_n)$.

(a) Prove that

$$G^*(dy_1 \wedge \cdots \wedge dy_n) = \det(DG)\, dx_1 \wedge \cdots \wedge dx_n.$$

(b) Let

$$P : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad P(r, \theta) = (r\cos\theta, r\sin\theta).$$

Compute $P^*(dx \wedge dy)$.

(c) Let

$$S : (0, \infty) \times (0, \pi) \times \mathbb{R} \longrightarrow \mathbb{R}^3$$

be the spherical-coordinate map

$$S(\rho, \varphi, \theta) = \big(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi\big).$$

Compute

$$S^*(dx \wedge dy \wedge dz)$$

as a multiple of $d\rho \wedge d\varphi \wedge d\theta$.

(d) Explain how the answer in part (c) changes if the last two parameters are written in the order (θ, φ) instead of (φ, θ).