Exam 1 – 02/16/05

Math 241H

Name: SOLUTIONS

*In your exam book, CLEARLY LABEL your solutions by number and part.*

1. Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j}$.

[8] (a) Are $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ parallel, perpendicular, or neither? Explain.

$\mathbf{a}\cdot(\mathbf{b} - \mathbf{c}) = 2 - 12 + 6 = -4 \neq 0$, hence the vectors are not perpendicular.

$\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = -17\mathbf{i} + \cdots \neq \mathbf{0}$, hence the vectors are not parallel. You can also see directly that $\mathbf{b} - \mathbf{c}$ is not a scalar multiple of $\mathbf{a}$, and this also shows the vectors are not parallel.

[10] (b) Find the distance from the endpoint of $\mathbf{a}$ to the line $\mathbf{r}(t) = \mathbf{b} + t\mathbf{c}$.

Take $P_0 = (1, 3, -2)$ (the end-point of $\mathbf{a}$). The direction vector of the line can be taken to be $\mathbf{L} = \mathbf{c} = \mathbf{i} + \mathbf{j}$. Take $P_1 = (3, -3, -3)$, a point on the line. Then $\overrightarrow{P_0P_1} = 2\mathbf{i} - 6\mathbf{j} - \mathbf{k}$, and

$$\mathbf{L} \times \overrightarrow{P_0P_1} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -6 & -1 \end{bmatrix} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}.$$

Thus,

$$D = \frac{\|\mathbf{L} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{L}\|} = \frac{\sqrt{66}}{\sqrt{2}} = \sqrt{33}.$$

[5] (c) Find the volume of the parallelpiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$$\text{vol} = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \left|\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right| = \left|\det \begin{bmatrix} 1 & 3 & -2 \\ 3 & -3 & -3 \\ 1 & 1 & 0 \end{bmatrix} \right| = 18.$$

[7] (d) Find the equation of the plane $\mathcal{P}$ whose normal is perpendicular to both **a** and **b** and which goes through the point $(0,1,1)$.

A choice for the normal vector is

$$\mathbf{N} = \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 3 & -3 & -3 \end{bmatrix} = -15\mathbf{i} - 3\mathbf{j} - 12\mathbf{k}.$$

We can also use $\mathbf{N} = 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. The equation of the plane is

$$5(x-0) + (y-1) + 4(z-1) = 0,$$

or

$$5x + y + 4z = 5.$$

[8] (e) Find the distance between the plane $\mathcal{P}$ in (d) and the point $(0,0,0)$.

Take $P_0 = (0,1,1)$ (a point in the plane), and $P_1 = (0,0,0)$. Then

$$D = \frac{|\mathbf{N} \cdot \overrightarrow{P_0P_1}|}{\|\mathbf{N}\|} = \frac{|-1-4|}{\sqrt{42}} = \frac{5}{\sqrt{42}},$$

since $\|\mathbf{N}\| = \|5\mathbf{i} + \mathbf{j} + 4\mathbf{k}\| = \sqrt{42}$.

[10] 2. If the following limit exists, explain why and compute it. If the limit does not exist, explain why not.

$$\lim_{t \to 1^+} (2 + e^{\frac{1}{1-t}})\,\mathbf{i} \ + \ \sqrt{t-1}\,\mathbf{j} \ + \ \ln(t)\,\mathbf{k}.$$

The limit does exist, because the limit of each component function exists. The main thing to check was the first component function. Observe that $\frac{1}{1-t} \to -\infty$ as $t \to 1^+$. Thus $e^{\frac{1}{1-t}} \to 0$ as $t \to 1^+$. Result: the limits of the other two component functions clearly being 0, we get the final limit to be $2\mathbf{i}$.

3. Suppose the velocity vector of a particle at time $t$ is given by $\mathbf{v}(t) = -e^{-t}\mathbf{i} + te^{t^2}\mathbf{j} + t\mathbf{k}$.
[10] (a) Find the formula for the position vector $\mathbf{r}(t)$, if $\mathbf{r}(0) = -\mathbf{k}$.

Integrating $\mathbf{v}(t) = \mathbf{r}'(t)$, we find $\mathbf{r}(t) = (e^{-t} + x_0)\mathbf{i} + (\frac{1}{2}e^{t^2} + y_0)\mathbf{j} + (\frac{1}{2}t^2 + z_0)\mathbf{k}$, for some constants $x_0, y_0, z_0$. Plug in $t = 0$ and set equal to $-\mathbf{k}$ to find the constants $x_0, y_0, z_0$. We get

$$\mathbf{r}(t) = (e^{-t} - 1)\mathbf{i} + (\frac{1}{2}e^{t^2} - \frac{1}{2})\mathbf{j} + (\frac{1}{2}t^2 - 1)\mathbf{k}.$$

[10] (b) Find the acceleration vector $\mathbf{a}(t)$.

Just differentiate $\mathbf{v}(t)$, to get $\mathbf{a}(t) = e^{-t}\mathbf{i} + (e^{t^2} + 2t^2e^{t^2})\mathbf{j} + \mathbf{k}$.

4. Let $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}$ be the parametrization of a curve.
[5] (a) Is this parametrization smooth? Explain.

Yes, because $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \sqrt{2}\mathbf{k}$ exists, is continuous, and is $\neq \mathbf{0}$, at every $t$.

[10] (b) Find the arclength of the curve traced out as $t$ ranges from $t = 1$ to $t = 2$.

$$\text{arclength} = \int_1^2 \|\mathbf{r}'(t)\| dt$$

which is

$$\int_1^2 (\sqrt{e^{2t} + e^{-2t} + 2})dt = \int_1^2 e^t + e^{-t}\, dt = [e^t - e^{-t}]_1^2 = e^2 - e^{-2} - e^1 + e^{-1}.$$

[10] (c) Find the curvature at $t = 0$.

The easiest way is to use the formula $\kappa(0) = \frac{\|\mathbf{v}(0)\times\mathbf{a}(0)\|}{\|\mathbf{v}(0)\|^3}$. Note that $\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$, so that

$$\mathbf{v}(0) \times \mathbf{a}(0) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 2\mathbf{k},$$

which has length $\sqrt{8} = 2\sqrt{2}$. Since $\|v(0)\|^3 = 8$, we see that the curvature is $1/\sqrt{8}$.

[7] 5. Suppose a particle moves in $\mathbb{R}^3$ in such a way that at all times, the velocity vector and the acceleration vectors are perpendicular. Prove that the speed of the particle is constant with respect to time.

Consider $\frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t)$. By using the product rule for differentiation of dot products of two functions, this is $\mathbf{v}(t) \cdot \mathbf{v}'(t) + \mathbf{v}'(t) \cdot \mathbf{v}(t) = 2\mathbf{v}(t) \cdot \mathbf{v}'(t)$, which is zero by hypothesis. Thus, $\|v(t)\|$, being the square-root of the function $\mathbf{v}(t) \cdot \mathbf{v}(t)$ which has vanishing derivative everywhere, is constant.