Quiz 3 – 02/23/05

Math 241H

Name:

1. (a) (4 points) Find the domain of the function $f(u, v) = \ln \frac{u^2 + v^2}{(u^2 - v^2)^2}$.

$\frac{u^2 + v^2}{(u^2 - v^2)^2}$ is always positive, and it is finite when $u \neq \pm v$.
Hence, the domain is $\{(u, v) | u \neq \pm v\}$.

(b) (4 points) Sketch the quadric surface $x = z^2 - 2$ in $\mathbb{R}^3$.

2. For parts (a) and (b), determine whether the limit exists, and if so, evaluate it. Explain your answers.
(a) (5 points) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$.

$0 \leq |x| \leq \sqrt{x^2 + y^2}$ and $0 \leq |y| \leq \sqrt{x^2 + y^2}$

=> $0 \leq |xy| \leq x^2 + y^2$

=> $0 \leq \frac{|xy|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2 + y^2}$

Since $\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0$, by the squeeze theorem, we get $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

(b) (2 points) $\lim_{(x,y)\to(1,1)} \frac{\cos^2(x^2 y \pi) + e^{x-y}}{x^2 + \sin(3y\pi/2)}$.

Does not exist because $x^2 + \sin(3y\pi/2) \to 0$

(c) (4 points) Is the following function continuous at $(0,0)$? Explain.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$f(x,y)$ is not continuous, since, along the line $y = x$ the function is equal to $1/2$ (ie. does not approach 0).

3. (a) (5 points) Verify that $z = \ln(x^2+y^2)$ satisfies Laplace's equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \qquad => \qquad \frac{\partial^2 z}{\partial x^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \qquad => \qquad \frac{\partial^2 z}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

Clearly $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

(b) (6 points) Let $u(x,t) = e^{ax^2+bxt+ct^2}$ for a, b, c positive constants. Verify that u satisfies the *wave equation* $\dfrac{\partial^2 u}{\partial x^2} = r^2 \dfrac{\partial^2 u}{\partial t^2}$ for $r = 1$ at all x, t only if $a = c$ and $b = 2a$. (Feel free to use the back.)

$$\frac{\partial u}{\partial x} = (2ax+bt)e^{ax^2+bxt+ct^2} \qquad \frac{\partial u}{\partial t} = (bx+2ct)e^{ax^2+bxt+ct^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \{2a + (2ax+bt)^2\}e^{ax^2+bxt+ct^2} \qquad \frac{\partial^2 u}{\partial^2 t} = \{2c + (bx+2ct)^2\}e^{ax^2+bxt+ct^2}$$

For $r = 1$ the wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$2a + (2ax+bt)^2 = 2c + (bx+2ct)^2$$

Which is true only if $a = c$ and $b = 2a$.