# Final Exam – 12/20/03

Math 340

Name:

1. True/False (Circle one; 3 points each, no partial credit).

**T** **F** (a) If $A$ is an $n \times n$ matrix and $\det(A) = 0$, then 0 is an eigenvalue for $A$.

True. $\det(A) = 0 \Rightarrow$ there exists $\mathbf{x} \neq 0$ such that $A\mathbf{x} = \mathbf{0}$. But then by definition $\mathbf{x}$ is an eigenvector with eigenvalue 0.

**T** **F** (b) Suppose $A$ is an $n \times n$ matrix. Suppose $\mathbf{x}, \mathbf{y}$ are non-zero vectors in $\mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, for scalars $\lambda \neq \mu$. Then $\mathbf{x} \cdot \mathbf{y} = 0$.

False. This would be true if $A$ were assumed symmetric. Counterexample: let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (so $\lambda = 1$), and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (so $\mu = 0$). Note that $\mathbf{x} \cdot \mathbf{y} = 1 \neq 0$.

**T** **F** (c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable everywhere. Then $f$ is integrable over any closed rectangle $A \subset \mathbb{R}^n$.

True. Any differentiable function is continuous, and any continuous function is integrable over any closed rectangle.

**T** **F** (d) Suppose $f$ is an integrable function on a closed rectangle $A \subset \mathbb{R}^n$ and $\int_A f = 0$. Then $f$ is identically zero.

False. We know that you can change a function at any finite number of points without changing the value of its integral. So for example, $f$ could be zero except at one point.

**T** **F** (e) Let $f$ be a continuous function on a closed and bounded subset $S$ of $\mathbb{R}^n$. Then there is some element $\mathbf{x} \in S$ such that $\sup\{f(\mathbf{y}) \mid \mathbf{y} \in S\} = f(\mathbf{x})$.

True. A general theorem stated in class asserts that any cont's function on a closed and bounded subset attains its maximum value. This implies that such an $\mathbf{x}$ exists.

2. (a) (10 points) Find the eigenvalues for the matrix

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

The characteristic polynomial is

$$\det \begin{bmatrix} x-2 & 0 & -1 \\ 0 & x-1 & 0 \\ -1 & 0 & x-2 \end{bmatrix} = (x-1)^2(x-3).$$

Therefore the eigenvalues are $\{1, 3\}$.

(b) (10 points) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for $B$.

We find the required basis by finding an orthonormal basis for each eigenspace. To do that, we first find a basis for each eigenspace, and then we use Gram-Schmidt to orthogonalize it, if necessary (that is, if it has dim $\geq 2$).

The eigenspace $E_3$ for the eigenvalue 3 is the kernel of the matrix $B-3I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, which has reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the solution set is given by: $x = z$ is free, and $y = 0$.

So $E_3$ has O.N. basis $\{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\}$.

The eigenspace $E_1$ is the kernel of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ which is row-equivalent to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So the solution set is given by: $-x = z$ is free and $y$ is free. So a O.N. basis for $E_1$ is $\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}\}$.

Hence, the desired O.N. basis of $\mathbb{R}^3$ consisting of eigenvectors for $B$ is

$$\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}\}.$$

3. (5 points) Suppose $z = f(x, y)$ is a differentiable function of $(x, y)$, and suppose $x = u + v$ and $y = u - v$. Show that

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 2\frac{\partial z}{\partial x}.$$

(Hint: Use the chain rule to write out $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as explicitly as you can.)

*Method 1.* We have

$$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.$$

Similarly, we get

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}.$$

Adding these two equations gives the result.

*Method 2.* By applying the chain rule to the composition of $f$ and $g(u,v) = (u+v, u-v) = (x(u,v), y(u,v))$ (thinking of everything as column vectors) we get

$$D(f \circ g)(u,v) = Df(g(u,v))\, Dg(u,v),$$

or in other words

$$\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x}(u,v) & \frac{\partial z}{\partial y}(u,v) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} & \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \end{bmatrix},$$

which yields the result upon adding the two entries of the $1 \times 2$ matrices on each side.

4. (15 points) Find the point on the sphere $x^2 + y^2 + z^2 = 3/4$ which is farthest from the point $(1, -1, 1)$.

*Set-up.* We maximize the square of the "distance to the point function", namely $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$, subject to the constraint $g = 0$, where $g(x,y,z) = x^2 + y^2 + z^2 - 3/4$. The Lagrange equation $\nabla f = \lambda \nabla g$ gives us (after cancelling factors of 2) the equations

$$\begin{aligned} (1-\lambda)x &= 1 \\ (1-\lambda)y &= -1 \\ (1-\lambda)z &= 1. \end{aligned}$$

*Find the solutions.*

Note that $\lambda \neq 0$, since $(1, -1, 1)$ does not lie of the sphere in question. Also, it is clear that $\lambda \neq 1$. so we get $x = \frac{1}{1-\lambda}$, $y = \frac{-1}{1-\lambda} = -x$, and $z = \frac{1}{1-\lambda} = x$. The only points of the form $(x, -x, x)$ which satisfy $g = 0$ are those with $x = \pm\frac{1}{2}$. So we get two possible solutions to the Lagrange equations and the constraint $g = 0$:

$$\{(1/2. - 1/2, 1/2), (-1/2, 1/2, -1/2)\}.$$

*Compare the distances for the various solutions to find the maximum.*

We have $f(1/2, -1/2, 1/2) = 3/4$ and $f(-1/2, 1/2, -1/2) = 27/4$. So the answer is:

$$(-1/2, 1/2, -1/2).$$

5. (a) (10 points) Find and classify the critical points of the function $f(x, y, z) = x^2 + xy + z^2 + y^2$.

We solve $\nabla f = \mathbf{0}$, ie the equations

$$\begin{aligned} 2x + y &= 0 \\ x + 2y &= 0 \\ 2z &= 0. \end{aligned}$$

The only solution is $(0, 0, 0)$.

Next we need to see whether this critical point is a local maximum or minimum, using the "2nd derivative test". That is, we check the determinants of the principle minors $D_1, D_2, D_3$ of the Hessian matrix at the point $(0, 0, 0)$, which is

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We see $D_1 = 2 > 0$, $D_2 = 4 - 1 = 3 > 0$, and $D_3 = \det(H) = 6 > 0$. Hence $(0, 0, 0)$ is a local minimum.

(b) (5 points) Find the global maximum and global minimum for the above function and explain your answers (if the global max or min does not exist, explain why not).

By completing the square for the expression $x^2 + xy + y^2$, we see that

$$x^2 + xy + y^2 + z^2 = (x + \frac{y}{2})^2 + \frac{3}{4}y^2 + z^2,$$

which is clearly always $\geq 0$, and has no maximum value. Also, since it takes the value of 0 at $(0, 0, 0)$, that point is not only a local mininum, but actually is also the global minimum.

(c) (5 points) In which direction does $f$ *decrease* most rapidly at the point $(1, -1, -1)$?

The direction is greatest *increase* at a point is the direction of the gradiant at that point. So the direction of greatest *decrease* is the *opposite direction* of the gradiant at that point. We have

$$\nabla f = (2x + y, 2y + x, 2z)$$

which at $(1, -1, -1)$ is $(1, -1, -2)$. So the desired direction is

$$(-1, 1, 2)$$

or (the corresponding unit vector)

$$(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}).$$

6. (5 points) Consider the cone having height $h$ and base radius $r$ (See picture on board). Write down an integral over a region in the $xy$-plane whose value is the volume of this cone. Do not evaluate the integral.

*Answer.*

$$\int_{x^2+y^2\leq r^2} h - (\frac{h}{r})(\sqrt{x^2+y^2})\, dA.$$

7. (a) (5 points) Transform the iterated integral

$$\int_0^1 \Big(\int_{y^{1/3}}^{y} e^{-x^2}\, dx\Big)\, dy$$

into an integral over a subset of the plane. Sketch this subset.

*Answer.*

$$-\int_R e^{-x^2}\, dA.$$

Note that the minus sign comes in because for $y$ belonging to the interval $[0,1]$, we have $y \leq y^{1/3}$, so that the inner integration is "in the wrong order", and must be reversed before we can write it as an integral over a region in the plane. Also, the region $R$ is the region contained in the unit square $[0,1]\times[0,1]$ which is below the line $y = x$ and is above the graph of $y = x^3$.

(b) (5 points) Write the integral as an iterated integral by reversing the order of integration.

*Answer.*

$$-\int_0^1 (\int_{x^3}^{x} e^{-x^2}\, dy\,)\, dx.$$

(c) (5 points) Compute the integral.

*Answer.*

$$-\int_0^1 \left(\int_{x^3}^{x} e^{-x^2}\, dy\right) dx = -\int_0^1 (x - x^3)e^{-x^2}\, dx.$$

The term $xe^{-x^2}$ has an anti-derivative so presents no problems. For the term $x^3e^{-x^2}$, use integration by parts, writing it as $u\,dv$, where

$$u = x^2$$

and

$$dv = xe^{-x^2}\, dx.$$

The final answer is:

$$-\frac{1}{2}e^{-1}.$$

8. (5 points + the value of impressing me right before I assign final grades.) Let $A$ be an $n \times n$ symmetric matrix. Prove that all eigenvalues of $A$ are positive if and only if the the associated quadratic form $Q_A(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ is positive definite (i.e., $Q_A(\mathbf{x}) > 0$ if $\mathbf{x} \neq 0$).

*Answer.*
"if": Suppose $\lambda$ is an eigenvalue, and $A\mathbf{x} = \lambda\mathbf{x}$, for $\mathbf{x} \neq \mathbf{0}$. Multiplying by $\mathbf{x}^t$, we get $\mathbf{x}^t\, A\mathbf{x} = \lambda\mathbf{x}^t\,\mathbf{x}$, or

$$\lambda = \frac{\mathbf{x}^t\, A\mathbf{x}}{\mathbf{x}^t\,\mathbf{x}} = \frac{Q_A(\mathbf{x})}{\mathbf{x}^t\,\mathbf{x}} > 0.$$

(This all makes sense since $\mathbf{x} \neq \mathbf{0}$ implies $\mathbf{x}^t\,\mathbf{x} > 0$.)

"only if": Suppose $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$, where $\{\mathbf{x}_i\}$ is an O.N. basis of eigenvectors for $A$: $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, for each $i$. Then we have

$$Q_A(\mathbf{x}) = \mathbf{x}^t\, A\mathbf{x} = \mathbf{x}\cdot A\mathbf{x} = \sum_i a_i^2\lambda_i.$$

Since at least one $a_i \neq 0$ (because $\mathbf{x} \neq \mathbf{0}$), this last quantity is $> 0$, which is the desired conclusion.

9. (a) (5 points) Write down the Jacobian of the following function at $(0,0)$. Is this function differentiable at $(0,0)$? Explain.

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

*Answer.* Along the $x$- and $y$-axes, the numerator $xy$ vanishes, and so using the *definition* of partial derivative in terms of limits, we easily get

$$Jf(0,0) = [0,0].$$

Now to see if the function is *differentiable* at $(0,0)$, it is not enough to know that the partial derivatives exist at this point. We must use the *definition* of derivative as a limit. Namely, we ask whether the following limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{xy/\sqrt{x^2+y^2} - 0 - 0}{\sqrt{x^2+y^2}}.$$

But this is

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

which does NOT exist (approaching the origin along the axes, you get the limit of 0, but along the line $x = y$, you get $1/2$). So the function is NOT differentiable at $(0,0)$.

(b) (5 points) Compute the 2nd order Taylor expansion at $(0,0)$ for the function $\sin(x-y)$.

The easiest way is to substitute $x-y$ for $z$ in the Taylor expansion for the function $\sin(z)$ at the point $z = 0$:

$$\sin(z) = z - \frac{z^3}{3!} + \cdots.$$

and so

$$\sin(x-y) = (x-y) + R_2$$

since clearly all the terms after the first one have degree in $x, y$ at least three. Therefore, the final answer is:

$$\sin(x-y) = x - y + R_2.$$