

Exam 2 – 04/13/12  
SOLUTIONS

Math 403

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Answer all problems in the exam books provided. Clearly label your work on each problem with the corresponding number (and letter). If solving problems out of numerical order, indicate this prominently. Give clear and complete explanations (proofs should contain words explaining what you are doing, not just an assortment of numbers or symbols!).

Good luck!

1. Calculate the order of each permutation below.

(a) [10 points]  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 8 & 6 & 9 & 5 & 10 & 1 & 3 & 7 & 2 \end{pmatrix}$ .

We write this as a product of disjoint cycles:  $(1\ 4\ 9\ 7)(2\ 8\ 3\ 6\ 10)$ . The order is  $\text{lcm}\{4, 5\} = 20$ .

(b) [10 points]  $(2\ 7\ 8)(2\ 8\ 9\ 1)(5\ 1\ 4\ 8\ 3)$ .

As a product of disjoint cycles:  $(1\ 4\ 9)(3\ 5\ 7\ 8)$ , so the order is  $\text{lcm}\{3, 4\} = 12$ .

2. (a) [?? points] Describe all the group homomorphisms  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ .

The group  $\mathbb{Z}$  is cyclic with generator 1, so  $\phi$  is completely determined by  $\phi(1)$  and any choice for  $\phi(1)$  determines a unique homomorphism given by  $\phi(n) = n\phi(1) \pmod{4}$ . So the four homomorphisms can be indexed as  $\phi_i$  determined by  $\phi_i(1) = i \pmod{4}$ , for  $i = 1, 2, 3, 4$ .

(b) [?? points] Determine which of the homomorphisms in (a) are surjective.

$\phi_i$  is surjective iff  $i$  is a generator for the cyclic group  $\mathbb{Z}/4\mathbb{Z}$ , which happens iff  $(i, 4) = 1$ . So  $\phi_1$  and  $\phi_3$  are surjective but  $\phi_2$  and  $\phi_4$  are not.

(c) [?? points] Determine the kernel of each of the homomorphisms in (a).

The kernel of  $\phi_i$  is the set of integers  $n$  such that  $ni \equiv 0 \pmod{4}$ . If  $i = 1, 3$ , this is  $4\mathbb{Z}$ . If  $i = 4$ , it is  $\mathbb{Z}$ . If  $i = 2$ , it is  $2\mathbb{Z}$ .

3. Consider the upper triangular matrix group  $B = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$  (NOTE: we need to require  $ad \neq 0$  too). Let  $T$  be the subgroup of diagonal matrices  $T = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R}^\times \right\}$ , and let  $U$  be the subgroup of strictly upper triangular matrices  $U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$ .

(a) [10 points] Show that  $U$  is normal in  $B$  but  $T$  is not normal in  $B$ .

First we check that  $U$  is normal in  $B$ . Note that the inverse of  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$  is  $\begin{bmatrix} a^{-1} & -b/ad \\ 0 & d^{-1} \end{bmatrix}$ .

Now fix  $\begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}$  and note that

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & -b/ad \\ 0 & d^{-1} \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & (-b + ab' + 1)/d \\ 0 & 1 \end{bmatrix}$$

which still belongs to  $U$ . Thus conjugating an element of  $U$  by any element of  $B$  yields an element of  $U$ ; hence  $U$  is normal in  $B$ .

Next we show that  $T$  is not normal in  $B$ . Note that the inverse of  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  is  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ .

Then we note that  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in T$  but

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

which is not in  $T$ . Thus,  $T$  is not normal in  $B$ .

(b) [10 points] Find the center of  $B$ .

For fixed  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in B$  and variable  $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in B$ , we have

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

iff

$$aY + bZ = bX + dY$$

for all  $X, Y, Z$ . This happens iff  $b = 0$  and  $a = d$ . So, the center of  $B$  is the set of matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

for  $a \in \mathbb{R}^\times$ .

4. [20 points] Determine all the Sylow subgroups of the group  $A_4$ .

2-Sylows: The order of  $A_4$  is  $12 = 2^2 \cdot 3$ . So  $n_2 = 1 + 2k|3$ . So  $n_2 = 1$  or  $n_2 = 3$ . One subgroup of order 4 is the Klein 4-group

$$\{e, (12)(34), (13)(24), (14)(23)\}.$$

It belongs to  $A_4$  since all its elements are even elements. We know that this subgroup is normal in  $S_4$ , hence also in  $A_4$ . It is a 2-Sylow, and all 2-Sylows are conjugate, so it is the only 2-Sylow since it is normal.

3-Sylow: We have  $n_3 = 1 + 3k|4$ . So,  $n_3 = 4$  if we find more than one subgroup of order 3. The element  $(123)$  is a product of 2 transpositions, so it is even and thus belongs to  $A_4$ . It generates a subgroup (a 3-Sylow)  $\langle(123)\rangle$ . We find the others in a similar way. There are four 3-Sylows:

$$\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle.$$

5. Let  $G$  be a group. We say an automorphism of  $G$  is *inner* if it is of the form  $c_g(x) := gxg^{-1}$  for some element  $g \in G$ . Denote by  $\text{Inn}(G)$  the set of inner automorphisms of  $G$ . This is a subset of  $\text{Aut}(G)$ , the group under composition of all automorphisms of  $G$ .

(a) [10 points] Prove that  $\text{Inn}(G)$  is a normal subgroup of  $\text{Aut}(G)$ .

We first show it is a subgroup. Given  $g_1, g_2 \in G$ , we note that

$$c_{g_1 g_2} = c_{g_1} \circ c_{g_2} \tag{1}$$

which is seen by calculating the effect of each side on  $x \in G$ : on the LHS we get  $(g_1 g_2)x(g_1 g_2)^{-1}$  and on the RHS we get  $g_1(g_2 x g_2^{-1})g_1^{-1}$ . These are clearly the same.

Equation (1) shows that  $\text{Inn}(G)$  is closed under multiplication (=composition) in  $\text{Aut}(G)$ . Also, the inverse of  $c_g$  is  $c_{g^{-1}}$ :  $c_g \circ c_{g^{-1}} = c_e = \text{id}$ . So  $\text{Inn}(G)$  is closed under taking inverses in the ambient group  $\text{Aut}(G)$ . Thus  $\text{Inn}(G)$  is a subgroup of  $\text{Aut}(G)$ .

Next we show  $\text{Inn}(G)$  is *normal* subgroup of  $\text{Aut}(G)$ . Let  $\theta \in \text{Aut}(G)$  and  $c_g \in \text{Inn}(G)$ . We see that

$$\theta \circ c_g \circ \theta^{-1} = c_{\theta(g)}$$

by evaluating both sides on  $x \in G$ : on the LHS we get  $\theta(g\theta^{-1}(x)g^{-1})$ , which because  $\theta$  is a homomorphism becomes  $\theta(g)x\theta(g)^{-1}$ , which is  $c_{\theta(g)}(x)$ .

This shows that  $\theta \text{Inn}(G) \theta^{-1} \subseteq \text{Inn}(G)$  for each  $\theta \in \text{Aut}(G)$ , which is what we needed to show.

(b) [5 points] Prove that  $G \rightarrow \text{Inn}(G)$  given by  $g \mapsto c_g$  is a surjective group homomorphism.

The equation (1) exactly shows that  $g \mapsto c_g$  is a homomorphism. The surjectivity follows from the fact that by definition, every inner automorphism is of the form  $c_g$  for some  $g \in G$ .

(c) [5 points] Prove that  $G/Z(G)$  is isomorphic to  $\text{Inn}(G)$ .

This follows immediately from (b) and the First homomorphism theorem, once we show that  $\ker[g \mapsto c_g]$  is  $Z(G)$ . But  $g \in \ker$  iff  $c_g = \text{id}$ , which happens iff  $c_g(x) = x$  for all  $x \in G$ , which happens iff  $g x g^{-1} = x$  for all  $x \in G$ . This happens in turn iff  $x \in Z(G)$ .