Some answers for Homework 12

Math 406

Here are solutions to some of the trickier problems :

Section 7.1, # 22: Show that if the equation $\phi(n) = k$ where k is a positive integer has exactly one solution n, then $36|n$.
Proof. We proceed in fours steps; first we show $2|n$, then $4|n$, then $3|n$, and finally $9|n$. At that point we will know $36|n$.

Assume n is odd. Then we have $\phi(2n) = \phi(2)\phi(n) = \phi(n)$. But this violates the hypothesis that n is the unique solution to $\phi(n) = k$. Thus we know n is even.

Suppose that $n = 2m$ where m is odd. Then as above we get $\phi(n) = \phi(m)$, again violating the hypothesis. This shows that $4|n$.

At this point we can write $n = 2^a m$, where m is odd and $a \geq 2$. Suppose that $(2, m) = (3, m) = 1$. Then we have $\phi(n) = 2^{a-1}\phi(m)$. But also, $\phi(2^{a-1} \cdot 3 \cdot m) = 2^{a-2} \cdot 2 \cdot \phi(m) = 2^{a-1}\phi(m)$, which violates the hypothesis on n since $n \neq 2^{a-1}3m$. Hence $3|n$.

Finally suppose that $n = 2^a 3m$, where $(2, m) = (3, m) = 1$. Then $\phi(n) = 2^a\phi(m) = \phi(2^{a+1}m)$, again violating the hypothesis since $n \neq 2^{a+1}m$. Thus $9|n$ and we are done.

Section 7.3, #14: Show that if $n = p^a q^b$, where p and q are distinct odd primes and a and b are positive integers, then n is deficient.
Proof. We need to show that $2p^a q^b > \sigma(p^a q^b)$. Using the fact that σ is multiplicative together with the formula for σ on prime powers, we see that we need to show

$$2p^a q^b > (\frac{p^{a+1} - 1}{p - 1})(\frac{q^{b+1} - 1}{q - 1}).$$

Clearing denominators this becomes

$$2(p - 1)(q - 1)p^a q^b > (p^{a+1} - 1)(q^{b+1} - 1).$$

Dividing both sides by $p^a q^b$, this becomes

$$2(p - 1)(q - 1) > (p - \frac{1}{p^a})(q - \frac{1}{q^b}).$$

To get a feel whether this is true, imagine letting a and b be arbitrarily large. Then if the above inequality were true, it would imply (by letting $a, b \to \infty$), that

$$2(p - 1)(q - 1) \geq pq.$$

Conversely, if we could establish this latter inequality, the former one would also follow, since no matter what a and b are, we certainly have $pq > (p - \frac{1}{p^a})(q - \frac{1}{q^b})$.

Now, to prove $2(p-1)(q-1) \geq pq$, we first transform this by dividing both sides by pq; we see we need to prove

$$2(1 - \frac{1}{p})(1 - \frac{1}{q}) \geq 1.$$

Now recall that without loss of generality $p > q$ and p, q are odd primes, so that $p > 4$ and $q \geq 3$. Thus the left hand side above is greater than

$$2(1 - \frac{1}{4})(1 - \frac{1}{3}) = 2(\frac{3}{4})(\frac{2}{3}) = 1.$$

We are done.