

# Homework 1 – due 09/05/07

## Math 600

1. (5 pts) Let  $(G, \cdot)$  be a set endowed with an associate binary operation, such that the following conditions hold:

- (i)  $\exists e \in G$  such that  $ex = x$  for all  $x \in G$ ;
- (ii)  $\forall x \in G, \exists x' \in G$  such that  $xx' = e$ .

Show, by constructing an explicit example, that  $(G, \cdot)$  need not be a group. Which axiom in the definition of group can fail?

2. (a) (5 pts) Suppose  $\phi : \mathbb{R} \rightarrow \mathbb{Z}$  is a group homomorphism (of groups under addition). Prove that  $\phi(x) = 0$  for all  $x \in \mathbb{R}$ .

(b) (5 pts) Recall that  $\mathbb{Z}$  and  $\mathbb{Q}$  are groups under addition. Show that they are not isomorphic.

(c) (5 pts) Are the groups  $\mathbb{R}^\times$  and  $\mathbb{R}_{>0}$  isomorphic (as groups under multiplication)? Prove your answer.

3. (a) (5 pts) Let  $f \in \text{Aut}(\mathbb{R})$ , the set of automorphisms of  $\mathbb{R}$  as an additive group. Suppose that  $f(1) = 1$ . Suppose  $f$  is also a continuous function. Show that  $f = \text{id}$ . Does this remain true if we omit the hypothesis “ $f$  is continuous”?

(b) (5 pts). Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function which satisfies  $f(x + y) = f(x) + f(y)$  and  $f(xy) = f(x)f(y)$ , for all  $x, y \in \mathbb{R}$ . Show that either  $f \equiv 0$ , or that  $f$  is the identity map.