Homework 10 – due 11/07/07

Math 600

49. (5 points) Suppose M is an R-module such that every non-empty set of finitely generated submodules of M has a maximal element. Show that M is Noetherian.

50. Let F be a field and $R = F[X, Y]$. Show that the ideal $I = (X, Y)$ in R is not free as an R-module.

51. (a) Let M be a Noetherian R-module and $u : M \to M$ a surjective R-module homomorphism. Prove that u is an isomorphism. Hint: Consider $\ker(u^n)$ for $n = 1, 2, \ldots$.
(b) Suppose M is Artinian, and $u : M \to M$ is injective. Show that u is an isomorphism.

52. Suppose V is a simple R-module. Prove that $\mathrm{Hom}_R(V, V)$, the set of all R-linear maps $V \to V$, is a field. (Let me remark that the commutativity of R is crucial here.)

53. Let K be a field and V a K-vector space with $\dim_K(V) = |\mathbb{Z}|$ (for example, we could take $V = K[X]$). Let $R = \mathrm{Hom}_K(V, V)$. Prove that $R \cong R \oplus R$ as R-modules.

54. (5 points) Suppose $f : V \to V$ is a homomorphism of the R-module V to itself. Suppose that $f^2 = f$. Prove that $V \cong \ker(f) \oplus \mathrm{im}(f)$ as R-modules.