

Solutions to Homework 10

Math 600, Fall 2007

49 (5 points) Suppose M is a (left) R -module such that every non-empty set of finitely generated submodules of M has a maximal element. Show that M is Noetherian.

We will show that every submodule N of M is finitely generated (abbreviated f.g.), thereby showing M is Noetherian. Define f.g. submodules $N_i \subset N$ recursively as follows. Let $N_0 = \{0\}$ and assume N_0 upto N_k are already defined. If $N_k = N$ then N is f.g. as required. Else, pick $x \in N - N_k$ and set $N_{k+1} = N_k + R \cdot x$. If N is not f.g. then we have a strictly increasing chain N_i of f.g. submodules of M . By hypothesis stated in the problem, this chain has a maximal element N_n contradicting the strictly increasing property of the chain. Thus, N has to be f.g.

50 (10 points) Let F be a field and $R = F[X, Y]$. Show that the ideal $I = (X, Y)$ in R is not free as an R -module.

We only assume that F is a (commutative) integral domain. Let $p = p(x, y)$ and $q = q(x, y)$ be two distinct (nonzero) polynomials with zero constant term. The equation $0 = q \cdot p + (-p) \cdot q$ shows that p and q are dependent over R . This shows that if I is a free R -module, then it has rank at most one. Now we show that this situation is also not possible. Indeed, suppose I is a free of rank one, $I = R \cdot f(x, y)$, for some nonzero $f(x, y) \in I$. Since $F[y]$ is an integral domain (because F is a domain), the degree in x of any polynomial in I is greater than or equal to the degree in x of $f(x, y)$. Therefore for y to be in I , we must have that $f(x, y)$ is a polynomial in y alone. Repeating the argument with the roles of x, y interchanged, we see that $f(x, y)$ must be a constant, but there are no nonzero constants in I , thus showing I cannot be a free R -module.

51 (10 points)

(a) Let M be a Noetherian R -module and $u : M \rightarrow M$ a surjective R -module homomorphism. Prove that u is an isomorphism. Hint: Consider $\ker(u^n)$ for $n = 1, 2, \dots$.

(b) Suppose M is Artinian, and $u : M \rightarrow M$ is injective. Show that u is an isomorphism.

Part a): Consider the ascending chain of submodules $(u^n)^{-1}\{0\}$. By ACC, this chain stabilizes at $(u^k)^{-1}\{0\}$ for some k . In particular $(u^k)^{-1}\{\text{Ker}(u)\} = (u^k)^{-1}\{0\}$. However the surjectivity of

$u^k : M \rightarrow M$ implies $(u^k)^{-1}\{\text{Ker}(u)\}/(u^k)^{-1}\{0\} \simeq \text{Ker}(u)$, and thus $\text{Ker}(u) = 0$ and u is an isomorphism.

Part b): Let $M_n = u^n(M)$ be a descending chain of submodules of M . By DCC, this chain stabilizes at $M_k = M_{k+1}$. Considering u^{k+1} as $u^k \circ u$ we get $u_k : M_1 \subset M \rightarrow M_{k+1} \subset M_k$. Since u^k is injective, we have $M/M_1 \cong M_k/M_{k+1} \cong \{0\}$ and hence $M_1 \cong M$ or u is an isomorphism.

52 (10 points): Suppose V is a simple R -module. Prove that $\text{Hom}_R(V, V)$, the set of all R -linear maps from $V \rightarrow V$, is a field. (Commutativity of R is crucial here).

A division ring satisfies all requirements of a field except that multiplication is not commutative.

Claim 1 : When V is a cyclic left R - module, then $\text{Hom}_R(V, V)$, is a division ring.

(Aside: The proof does not use the commutativity of R , so we work with a general ring R in what follows. However, the noncommutativity of R has tricky consequences. For a left R -module M , the ring $\text{Hom}_R(M, M)$ is not a left R -module. Also, the maps $m \mapsto r \cdot m$ are not in $\text{Hom}_R(M, M)$)

Proof of Claim 1: Since V is a simple left R -module, given a nonzero element $f \in \text{Hom}_R(V, V)$, we know that its image is V and its kernel is zero. Thus f is an isomorphism and thus has a 2-sided inverse. Thus $\text{Hom}_R(V, V)$ is a division ring.

We next address the question, when $\text{Hom}_R(V, V)$ is a field. Those who are only interested in the case when R is commutative may ignore the noncommutative aspects of what follows. A simple R -module has to be cyclic, so let $V = R \cdot v$ for some $v \in V$. For the moment, we focus on the cyclic module $V = R \cdot v$ forgetting that it is a simple module. Let $I = \text{Ann}(v) = \{r \in R \mid r \cdot v = 0\}$. We have, I is a submodule of the left R -module R . Consider the map from the quotient left R -module $R/I \rightarrow V$ that takes $r + I \mapsto r \cdot v$. It is easy to check that this map is an isomorphism of left R -modules. So we now identify V with R/I .

Claim 2: Let S be the largest subring of R containing I such that I is a 2-sided ideal of S .

$\text{Hom}_R(R/I, R/I) \cong (S/I)^{\text{opp}}$ as rings.

(In the opposite ring the product $a * b$ is defined to be ba instead of ab)

Combining both the claims, we see that when R/I is a simple left R -module (i.e. I is maximal among left ideals) and S/I is commutative, then $\text{Hom}_R(R/I, R/I) = \text{Hom}_R(V, V)$ is a field. In particular when R is commutative and I is a maximal ideal, then $R/I = \text{Hom}_R(R/I, R/I) = \text{Hom}_R(V, V)$ is a field.

Proof of Claim 2: Let $f \in \text{Hom}_R(R/I, R/I)$. Let $f(1 + I) = x + I$, for some $x + I$. This completely determines f by R -linearity. There is just one restriction on what x can be: To see this, note that $f(r + I) = r \cdot f(1 + I) = rx + I$. This has to depend only on the cosets $r + I$ and $x + I$, which forces $Ix \subset I$, or that $x + I \in S/I$. Thus we have a bijection of sets $\text{Hom}_R(R/I, R/I) \rightarrow S/I$ taking $f \mapsto f(1 + I)$, which is easily seen to be a ring isomorphism: $\text{Hom}_R(R/I, R/I) \cong (S/I)^{\text{Opp}}$

53 (10 points): Let K be a field and V , a K -vector space with $\dim_K(V) = |\mathbb{Z}|$ (for example, we could take $V = K[X]$). Let $R = \text{Hom}_K(V, V)$. Prove that $R \cong R \oplus R$ as R -modules.

NOTE: We have to show that $R \cong R \oplus R$ as R -modules, not just as K -modules.

Any R -module homomorphism from $\Phi : R \oplus R \rightarrow R$ is determined (R -linearly) by $f_1 := \Phi(1, 0)$ and $f_2 := \Phi(0, 1)$. Let $\{e_i \mid i \in \mathbb{N}\}$ be a K -basis for V . Define f_1 by $e_{2i} \mapsto e_i$ and $e_{2i-1} \mapsto 0$ and define f_2 by $e_{2i-1} \mapsto e_i$ and $e_{2i} \mapsto 0$. Similarly define $r : e_i \mapsto e_{2i}$ and $s : e_i \mapsto e_{2i-1}$. We have $r \cdot f_1 + s \cdot f_2 = 1$ whence Φ is surjective. To show Φ is injective consider the equation $g_1 \cdot f_1 + g_2 \cdot f_2 = 0$. By considering the action of $g_1 \cdot f_1 + g_2 \cdot f_2 = 0$ on e_{2i} and e_{2i-1} we must have $g_1(e_i) = 0$ and $g_2(e_i) = 0$. Thus g_1, g_2 are zero and hence Φ is injective.

54 (5 points): Suppose $f : V \rightarrow V$ is a homomorphism of the R -module V to itself. Suppose that $f^2 = f$. Prove that $V \cong \ker(f) \oplus \text{im}(f)$ as R -modules.

Observe that $f|_{\text{im}(f)} = id$ and therefore $\ker(f) \cap \text{im}(f) = 0$. Together with the fact that the abelian group V is the (disjoint) union of the cosets of the subgroup $\ker(f)$ and that every coset has a representative from $\text{im}(f)$, we know that $\ker(f) \oplus \text{im}(f)$ has to be all of V .