

Homework 11 – due 11/14/07

Math 600

55. Let R be any commutative ring with identity, and let $S \subset R$ be a multiplicative subset. Prove the exactness of the functor $M \mapsto S^{-1}M$ from the category of R -modules to the category of $S^{-1}R$ -modules. In other words, if $M' \rightarrow M \rightarrow M''$ is an exact sequence of R -modules, then $S^{-1}M' \rightarrow S^{-1}M \rightarrow S^{-1}M''$ is an exact sequence of $S^{-1}R$ -modules.

56. Dummit-Foote, 12.1, # 2.

57. Dummit-Foote, 12.1, #4.

58. Dummit-Foote, 12.1, #11.

59. Let R be a PID. Let V be a finitely-generated torsion R -module. For each irreducible factor p_j ($1 \leq j \leq r$) of the annihilator of V , let V_j denote the submodule of V consisting of the elements killed by some power of p_j . Show that

$$V = \bigoplus_{j=1}^r V_j.$$

60. Let R be a commutative ring with identity. Let $A \in M_n(R)$ be such that $\det(A) = 0$. Show that there exists $\mathbf{x} \in R^n$ with $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$. HINT: We may suppose $A \neq 0$. Let k with $2 \leq k \leq n$ be the maximal integer such that some $(k-1) \times (k-1)$ minor A_{k-1} has non-zero determinant. Without loss of generality, we may assume the minor A_{k-1} is the $(k-1) \times (k-1)$ submatrix in the upper left corner of A . Let A_k be the $k \times k$ submatrix in the upper-left corner of A . Now apply Cramer's Rule to A_k (see Theorem 29 on p. 439 of Dummit-Foote).