Solutions to Homework 11

Math 600, Fall 2007

55 (10 points) Let R be any commutative ring with identity, and let $S \subset R$ be a multiplicative subset. Prove the exactness of the functor $M \mapsto S^{-1}M$ from the category of R-modules to the category of $S^{-1}R$-modules. In other words, if $M' \to M \to M''$ is an exact sequence of R-modules, then $S^{-1}M' \to S^{-1}M \to S^{-1}M''$ is an exact sequence of $S^{-1}R$-modules.

Let us denote by $i : M' \to M$ and $j : M \to M''$ the given maps with $\ker(j) = \mathrm{im}(i)$. Let $i_1 : S^{-1}M' \to S^{-1}M$ and $j_1 : S^{-1}M \to S^{-1}M''$. We need to show $\ker(j_1) = \mathrm{im}(i_1)$. Recall that every element of $S^{-1}M$ is of the form $s^{-1}m$ for some $s \in S$ and some $m \in M$, and that $m = 0$ in $S^{-1}M$ iff there exists some $t \in S$ with $tm = 0 \in M$. Therefore:

$$s^{-1}m \in \ker(j_1) \Leftrightarrow s^{-1}j(m) = 0 \Leftrightarrow j(m) = 0 \in S^{-1}M \Leftrightarrow tj(m) = 0 \in M \Leftrightarrow j(tm) = 0$$

for some $t \in S$. Using the fact that $\ker(j) = \mathrm{im}(i)$, we have some $m' \in M'$ with:

$$j(tm) = 0 \Leftrightarrow tm = i(m') \Leftrightarrow s^{-1}m = i_1(s^{-1}t^{-1}m') \Leftrightarrow s^{-1}m \in \mathrm{im}(i_1)$$

Thus $\ker(j_1) = \mathrm{im}(i_1)$.

Remark: Let $0 \to N \to M \to P \to 0$ be a short exact sequence of R-modules, then using the preceding result (thrice), the reader may show that, $0 \to S^{-1}N \to S^{-1}M \to S^{-1}P \to 0$ is a short exact sequence of $S^{-1}R$-modules. We use this observation later in these solutions.

56 Dummit-Foote, 12.1 #2 (10 points) Let R be a domain, M an R-module
a) If $\mathrm{rank}(M) = n$ and $\{x_1, \cdots x_n\}$ is max'l linearly independent set, and $N = Rx_1 + \cdots + Rx_n$, then show that $N \simeq R^n$ and M/N is a torsion module.

The R-module homomorphism from $R^n \to N$ sending the i^{th} standard basis vector of R^n to x_i is clearly surjective and the injectivity of this map is equivalent to the linear independence of $\{x_1, \cdots x_n\}$. Thus $N \simeq R^n$. Since $\mathrm{rank}(M) = n$, for any $m \in M$, the set $\{m, x_1, \cdots x_n\}$ is linearly dependent and thus some scalar multiple (which must be nonzero) of m is the submodule N, but this is equivalent to M/N being torsion.

b) If $N \subset M$ is a submodule with $N \simeq R^n$ and M/N is a torsion module, then $\text{rank}(M) = n$

Clearly $\text{rank}(M) = l \geq n$. If $\{x_1, \cdots x_l\}$ is a max'l linearly independent set, then for any $r_1, \cdots r_l \in R - \{0\}$, so is $\{r_1x_1, r_2x_2, \cdots, r_lx_l\}$ (Indeed, a linear dependence relation of the latter set is also one for the former set). Using the fact that M/N is torsion, pick $r_i \in R - \{0\}$ such that $r_ix_i \in N$. The fact that $\text{rank}(N) = n$ together with the linear independence of the set $\{r_1x_1, r_2x_2, \cdots r_lx_l\}$ now implies $l \leq n$, whence $l = n$.

57 Dummit-Foote, 12.1 #4 (10 points) Let R be a domain, $N \subset M$ are R-modules, with $\text{rank}(M) = n$ and $\text{rank}(N) = r$, and $\text{rank}(M/N) = s$, then show that $n = r + s$

Proof 1: Let $\{x_1, x_2, \cdots, x_s, y_1, y_2, \cdots y_r\}$ be such that the $\{x_1 + N, \cdots, x_s + N\}$ is a max'l linearly independent set of M/N and $\{y_1, y_2, \cdots, y_r\}$ is a max'l linearly independent set of N. It is easy to check that S is linearly independent, hence $n \geq r + s$. Now let $S = \{z_1, z_2, \cdots, z_n\}$ be a max'l linearly independent set of M, then as in P56 b), so is $S' = \{z_1, \cdots z_s, r_1z_{s+1}, \cdots, r_{n-s}z_n\}$ for any $r_1, \cdots r_{n-s} \in R - \{0\}$. Since $\{z_1 + N, \cdots z_s + N, z_{s+i} + N\}$ is dependent in M/N, pick $r_1, \cdots r_{n-s}$ such that $r_iz_{s+i} - w_i \in N$ where w_i is a linear combination of $\{z_1, \cdots z_s\}$. Let $n_i := r_iz_{s+i} - w_i$. By definition of the w_i, it is evident that the linear independence of S' is equivalent to the linear independence of $S'' = \{z_1, \cdots z_s, n_1, \cdots n_{n-s}\}$. The fact that $\text{rank}(N) = r$ together with the linear independence of $\{n_1, \cdots, n_{n-s}\}$ implies $n - s \leq r$ or $n \leq r + s$ whence $n = r + s$.

Proof 2: We now cast the preceding proof in slightly different terms, to make it neater. Let F denote the fraction field of R. We have $F = S^{-1}R$ where $S = R - \{0\}$. The reader may supply the proofs of the following simple facts, which put together yield $n = r + s$ as required.

Fact 1: $\text{rank}(M) = \dim_F(S^{-1}M)$

Fact 2: $S^{-1}(M/N) \simeq S^{-1}M/S^{-1}N$. (By remark after Problem 55)

Fact 3: For vector spaces $W \subset V$, we have: $\dim(V) = \dim(W) + \dim(V/W)$

58 Dummit-Foote, 12.1 #11 (10 points) Let R be a PID, $a \in R - \{0\}$. Let $M = R/(a)$. Let $p \in R$ be a prime, and let n be the largest power of p dividing a. Prove that:

$$p^{k-1}M/p^kM = \begin{Bmatrix} R/(p) & \text{if } \ k \leq n \\ 0 & \text{if } \ k > n \end{Bmatrix}$$

Let $I = (p^{k-1}, a)$ and $J = (p^k, a)$ be ideals of R. Consider the surjective R-module homomorphism $f : I \to p^{k-1}M$ which is the restriction of $R \to R/(a) = M$ to I. We have, by the isomorphism theorems, $I/J \simeq p^{k-1}M/p^kM$ as R-modules. Let $\mu = \gcd\{p^{k-1}, a\}$ and $\nu = \gcd\{p^k, a\}$. We have $I = \mu R$ and $J = \nu R$ and by the isomorphism theorem applied to the surjective map $R \to \mu R$ taking $1 \mapsto \mu$, we get $I/J = R/(\nu/\mu)$. Finally ν/μ is p if $k \leq n$ and 1 if $k > n$. Hence we obtain the result.

59 (10 points): Let R be a PID. Let V be a finitely-generated torsion R-module. For each irreducible factor p_j $(1 \leq j \leq r)$ of the annihilator of V, let V_j denote the submodule of V consisting of the elements killed by some power of p_j . Show that $V = \oplus_{j=1}^{r} V_j$

This is obvious from the elementary divisor form of the structure theorem of f.g. modules over PIDs.

60 (10 points) Let R be a commutative ring with identity. Let $A \in M_n(R)$ be such that $\det(A) = 0$. Show that there exists $x \in R^n$ with $x \neq 0$ and $Ax = 0$.

Let $I_l \subset R$ be the ideal generated by all $l \times l$ minor determinants of A. By definition of the determinant we have $I_1 \supset I_2 \supset \cdots \supset I_n$. Since $A \neq 0$, we have $I_1 \neq 0$, and since $\det(A) = 0$, we have $I_n = 0$. Hence there is a largest k with $2 \leq k \leq n$ such that $I_{k-1} \neq 0$ and $I_l = 0$ for $l \geq k$. In our proof, we will transform $A \mapsto QAP^{-1}$ for some $Q, P \in GL(n, R)$ and we will require the definition of k to hold. Indeed, the minor determinants change, but the ideals stay: it is easy to check that I_1 (and also I_n) is unaffected by this transformation. For the I_l in between this is not so obvious, but it can be shown if we write down the minor determinants of QAP^{-1} and rearrange the terms carefully. Another conceptual way to see that the I_l are truly invariants of a R-module homomorphism $T : V \to W$ where V and W are free of rank n is to observe that I_l is the intersection of all ideals I such that the induced map from $\wedge^l V / \wedge^l (IV) \to \wedge^l W / \wedge^l (IW)$ is zero, where the $\wedge^l V$ are the l^{th} exterior powers of V. (For example $\wedge^1 V = V$ so the reader can check that the two definitions of I_1 agree).

Since $I_{k-1} \neq 0$, there is a $k-1 \times k-1$ minor of A which has nonzero determinant. By making both row and column movements, we move this minor to the $k-1 \times k-1$ minor on the top left. The matrix A changes to QAP^{-1} for some permutation matrices $Q, P \in GL(n, R)$. Denote the $k \times k$ minor on the top left by A_k. Note that all $k \times k$ minors of this rearranged matrix have zero determinant by using that fact that the ideal I_k is unaffected by $A \mapsto QAP^{-1}$. Let x_k be the

k^{th} column of $\text{adj}(A_k)$ and let $x = (x_k, 0, \cdots, 0)^t \in R^n$. When $l \geq k$, the l^{th} entry of the vector Ax is the determinant of the minor on rows $1, 2, \cdots, k-1, l$ and columns $1, 2, \cdots k$ and hence is zero. When $l < k$, the l^{th} entry of Ax is the $(l,k)^{\text{th}}$ entry of A_k $\text{adj}(A_k)$. Note that A_k $\text{adj}(A_k) = 0$ by Cramer's rule and using the fact that $\det(A_k) = 0$. Also the k^{th} entry of x is the nonzero minor determinant that we have singled out, hence we have obtained a solution to $Ax = 0$ with $x \neq 0$.

Another Proof: We now present a simple and conceptual proof (without details) of the result of this problem which requires the knowledge of exterior powers of a module (to be treated in MATH601). The n^{th} exterior power of R^n denoted $\wedge^n R^n$ is free of rank 1. The standard basis of R^n singles out a standard basis vector for $\wedge^n R^n \simeq R$. Any R-module homomorphism $A : R^n \to R^n$ induces a homomorphism $\wedge^n A : \wedge^n R^n \to \wedge^n R^n$ which must simply multiply the standard basis vector by a scalar: this scalar turns out to be $\det(A)$. Since $\det(A) = 0$, we know $\text{im}(\wedge^n A) = 0$. On the other hand it can be shown that $\text{im}(\wedge^n A) = \wedge^n W$, where $W = \text{im}(A)$. Thus $\wedge^n W = 0$. Suppose $\ker(A)$ were $\{0\}$ then $W = \text{im}(A) \simeq R^n$ and hence $\wedge^n W \simeq \wedge^n R^n \simeq R$ contradicting $\wedge^n W = 0$. Thus $\ker(A)$ cannot be zero, as was to be shown.

Remark

Let A be an $m \times n$ matrix with entries in R. If $m < n$ then we can add $n - m$ extra rows at the bottom of A to obtain A' with $\det(A') = 0$. By the result of this problem, there is an x with $A'x = 0$, whence $Ax = 0$. Now suppose $m \geq n$ and all $n \times n$ minors of A have zero determinant, then either proof presented above work perfectly well (almost unchanged) to show $\ker A \neq \{0\}$. Hence we summarize:

Let A be an $m \times n$ matrix with entries in a commutative ring R. If $m < n$ then A always has a nontrivial kernel. If $m > n$ and all $n \times n$ minors of A have zero determinant then again A has a nontrivial kernel.